Tuesday
10/1

<u>Ex:</u> $\mathbb{Z}\times\mathbb{Z}$ is not cyclic.

<u>proof:</u> Pick some $(a,b)\in\mathbb{Z}\times\mathbb{Z}$.

Suppose $\langle(a,b)\rangle=\mathbb{Z}\times\mathbb{Z}$.

<u>Claim:</u> $a=\pm1$ and $b=\pm1$.

Since $(1,1)\in\mathbb{Z}\times\mathbb{Z}$, then $(1,1)$ is generated by (a,b).

So, $(1,1)=(ka,kb)$ where $k\in\mathbb{Z}$.

So, $\boxed{1=ka}$ and $\boxed{1=kb}$

Since $k,a,b\in\mathbb{Z}$ the only possibilities are

$k=$

And

$\langle(1,1)\rangle=\langle$

So the

Theorem

Then

$\Rightarrow k=a=b=1$ or $k=a=b=-1$.

And

$\langle(1,1)\rangle=\langle(-1,-1)\rangle=\{\ldots,(-3,-3),(-2,-2),(-1,-1),(0,0),(1,1),(2,2),(3,3),\ldots\}\neq \mathbb{Z}\times\mathbb{Z}$

So there is no generator of $\mathbb{Z}\times\mathbb{Z}$. $\boxed{/\!/}$

d by

(a,b).

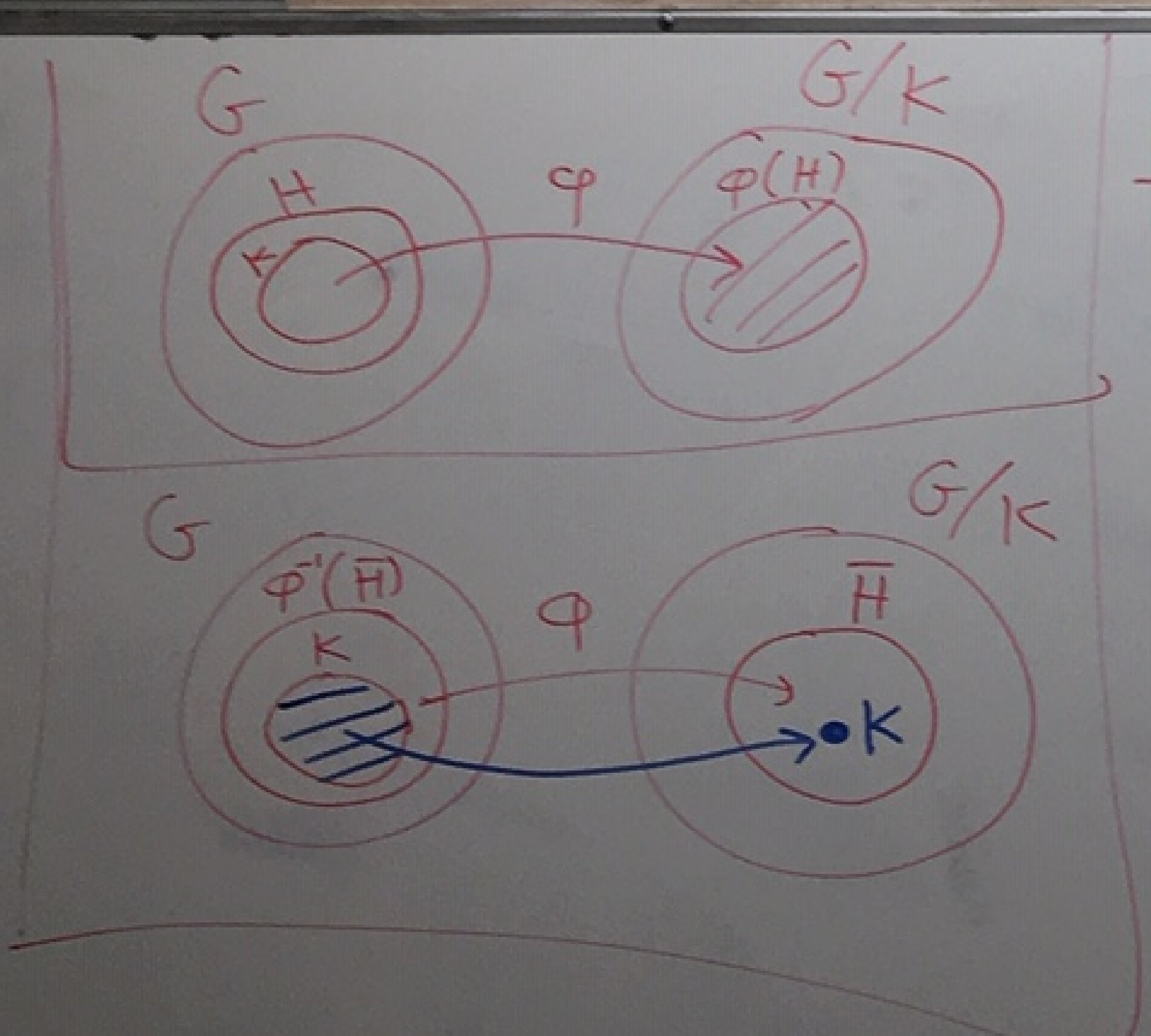
Theorem: Let G be a group and $K \trianglelefteq G$. Let $\varphi: G\to G/K$ be given by $\varphi(g)=gK$.

Then φ gives a one-to-one correspondence between the following sets:

$$\{H \mid H\leq G \text{ and } K\leq H\} \xleftrightarrow{\varphi} \{\bar{H} \mid \bar{H}\leq G/K\}$$

$$H \longmapsto \varphi(H)$$

$$\varphi^{-1}(\bar{H}) \longleftarrow\!\!\mapsto \bar{H}$$



proof: Given $H \le G$ we know from the lemma from last time that $\varphi(H) \le G/K$. Also from the lemma, if $\overline{H} \le G/K$ then $\varphi^{-1}(\overline{H}) \le G$. Is

$K \le \varphi^{-1}(\overline{H}) = \{ g \in G \mid \varphi(g) \in \overline{H} \}$.

Since $\overline{H} \le G/K$ we know the identity of G/K is in $\overline{H}$. That is, $K \in \overline{H}$.

Note if $g \in K$, then $\varphi(g) = gK \underset{\boxed{g \in K}}{=} K$.

So, for every $g \in K$, $\varphi(g) = K \in \overline{H}$.

So, $K \subseteq \varphi^{-1}(\overline{H})$.

Thus, $\boxed{\text{if } \overline{H} \le G/K, \text{ then } \varphi^{-1}(\overline{H}) \le G \text{ and } K \le \varphi^{-1}(\overline{H})}$.

The correspondence is one-to-one.

Suppose H_1, H_2 with $K \leq H_1 \leq G$ and $K \leq H_2 \leq G$ and $\varphi(H_1) = \varphi(H_2)$.

Let's show that $H_1 = H_2$.

Let's show $H_1 \subseteq H_2$.

Pick $h_1 \in H_1$.

Note that $\varphi(h_1) = h_1 K$.

Also since $\varphi(H_1) = \varphi(H_2)$ and $h_1 \in H_1$ and $\varphi(h_1) \in \varphi(H_1)$

we know $\varphi(h_1) \in \varphi(H_2)$.

So, $\varphi(h_1) = \varphi(h_2)$ for some $h_2 \in H_2$.

Thus, $h_1 K = h_2 K$.

Thus, $h_1 \in h_2 K$.

So, $h_1 = h_2 k$ where $k \in K$.

From above we know $K \leq H_2$

and thusly $k \in H_2$.

Since $h_2, k \in H_2$ we have

$h_1 = h_2 k \in H_2$,

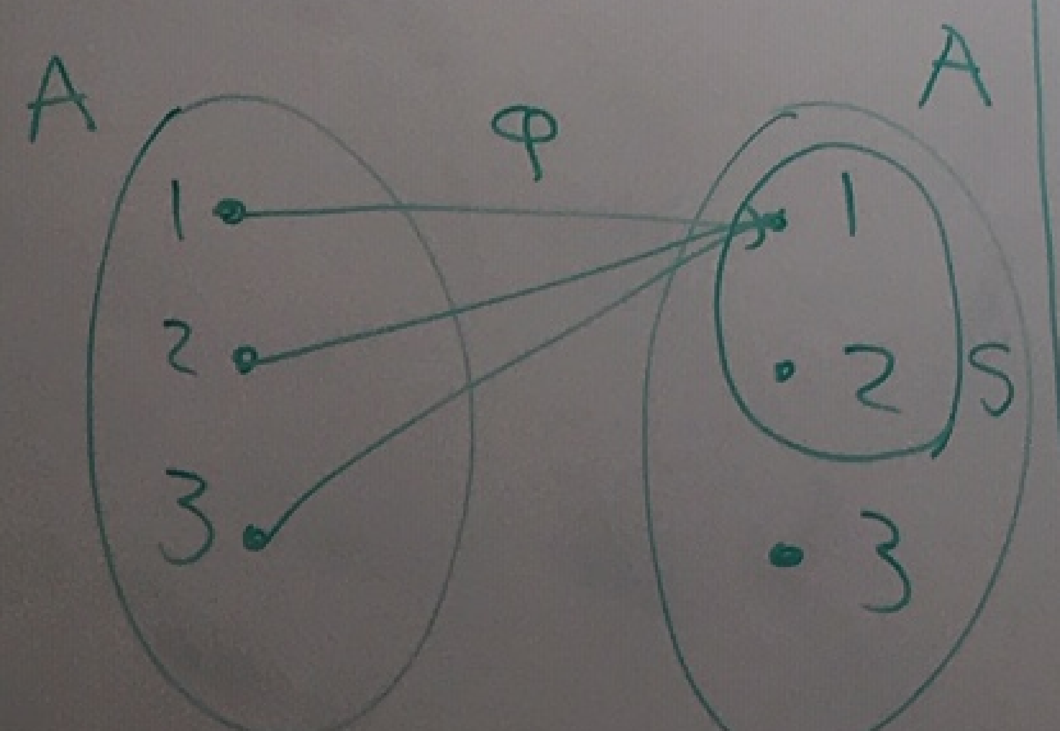
Therefore, $H_1 \subseteq H_2$,

Similarly, $H_2 \subseteq H_1$. So, $H_1 = H_2$.

Why can't use $\varphi(\varphi^{-1}(S))=S$

$\varphi : A \to A,\ S \subseteq A$

$\varphi(\varphi^{-1}(S)) \subseteq S$



$\varphi(\varphi^{-1}(S)) = \varphi(\{1,2,3\})$
$= \{1\} \subseteq S$

The correspondence is onto

Let $\overline{H} \leq G/K$.

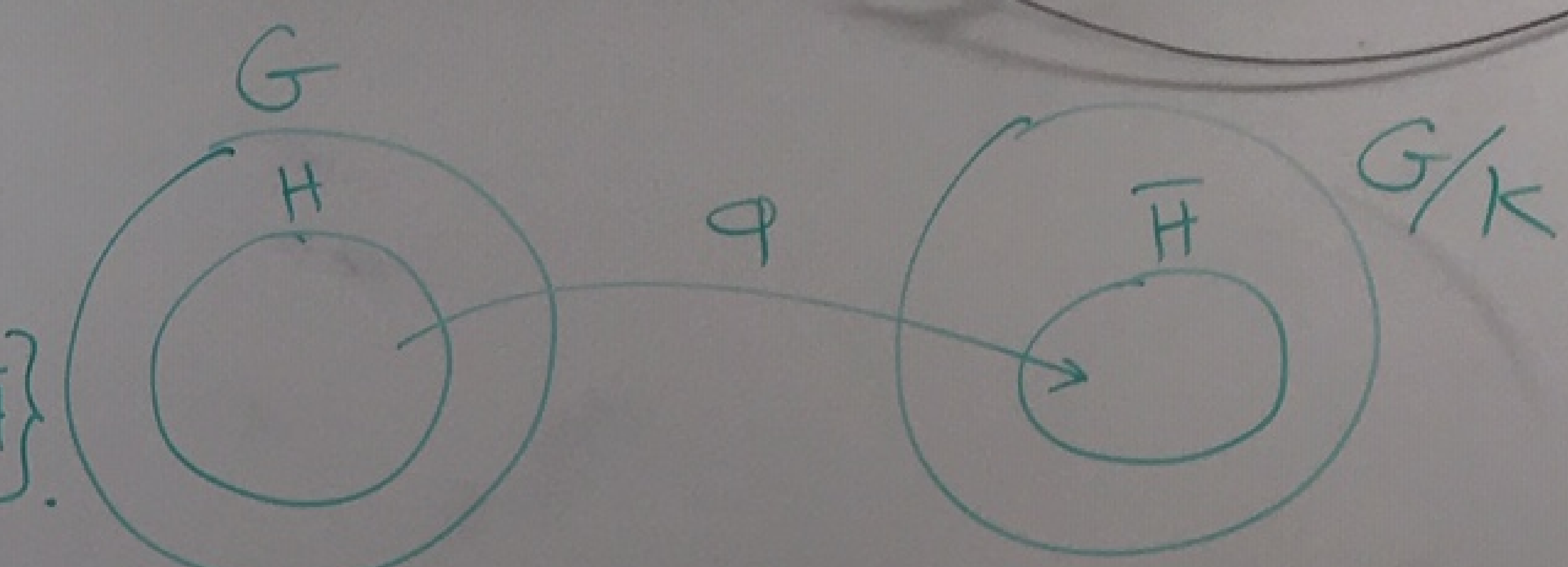
Let $H = \varphi^{-1}(\overline{H}) = \{g \in G \mid \varphi(g) \in \overline{H}\}$.

Two boards ago we showed that $K \leq H \leq G$.

Claim: $\varphi(H) = \overline{H}$.

If we prove this claim we are done.

Let $x \in \varphi(H)$.
Then $x = \varphi(g)$ where $g \in H = \varphi^{-1}(\overline{H})$.
Thusly, $x = \varphi(g) \in \overline{H}$.
So, $\varphi(H) \subseteq \overline{H}$.



Let $y \in \overline{H}$.
Then $y = gK$ where $g \in G$.
Note $\varphi(g) = gK \in \overline{H}$.
So, $g \in H$.
Therefore, $y = gK = \varphi(g) \in \varphi(H)$.
So, $\overline{H} \subseteq \varphi(H)$.
Therefore, $\varphi(H) = \overline{H}$. $\blacksquare$

Method 2 of claim

φ is onto.

It follows that

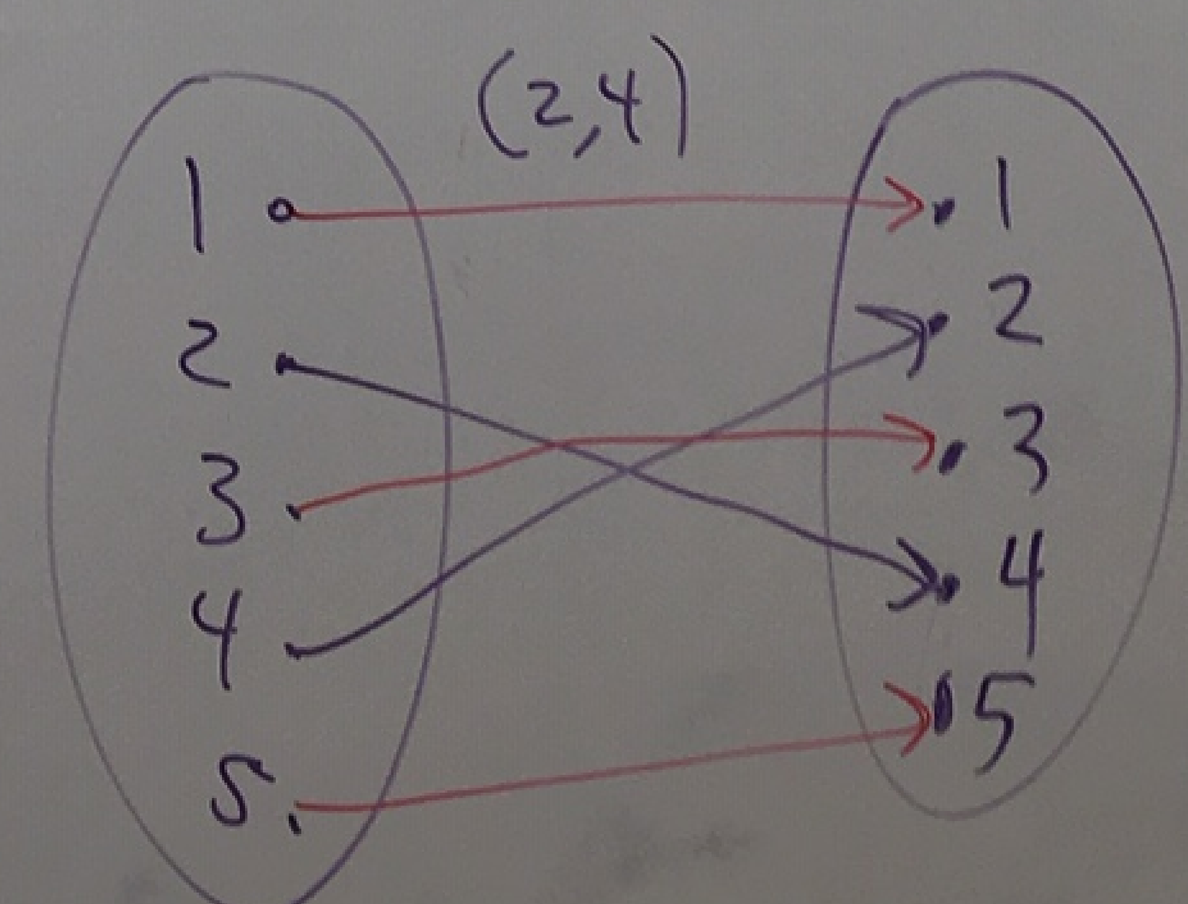
$$\varphi(H) = \varphi(\varphi^{-1}(\overline{H})) = \overline{H}.$$

$\boxed{\text{In general, if } f: A \to B \text{ and } f \text{ is onto } B \text{ and } S \subseteq B \text{ then } f(f^{-1}(S)) = S.}$

3.5 – Transpositions and the alternating group

Def: A 2-cycle in S_n is called a transposition

Ex: $(2,4) \in S_5$ is a transposition.



Proposition Any element of S_n can be written as a product of transpositions.

Method: Given $\tau \in S_n$, write τ as the product of disjoint cycles. Then use the following formula on each cycle:

$$(a_1, a_2, \ldots, a_n) = (a_1, a_m)(a_1, a_{m-1}) \cdots (a_1, a_2)$$

Ex:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 5 & 1 & 3 & 7 & 9 & 8 \end{pmatrix}$$

$$= (1,2,4,5)(3,6)(7)(8,9)$$

$$= (1,5)(1,4)(1,2)(3,6)(8,9)$$

$\boxed{\tau \text{ is the product of } 5 \text{ transpositions}}$

6 7 8 9
3 7 9 8

(3,6)(7)(8,9)

,2)(3,6)(8,9)

of
tions

Theorem: For any $\tau \in S_n$ there may be many ways of writing τ as a product of transpositions; however, if you express τ in two different ways as the product of transpositions then you can't have one way be an odd number of transpositions and the other way be an even number of transpositions.