10/22
Tuesday

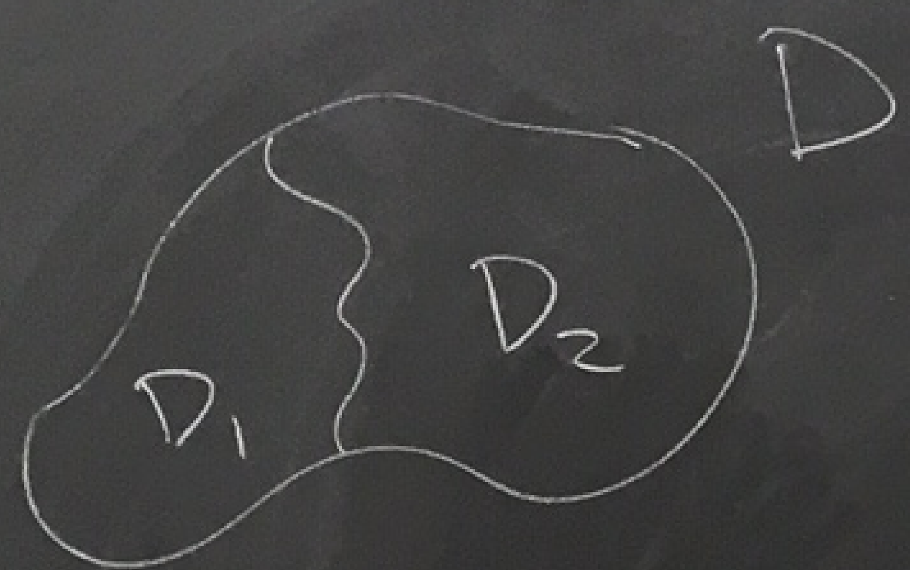
Some facts from 13.2

- $$\iint_D \left[f(x,y) \pm g(x,y)\right] dA = \iint_D f(x,y)\,dA \pm \iint_D g(x,y)\,dA$$

- $$\iint_D c f(x,y)\,dA = c \iint_D f(x,y)\,dA$$

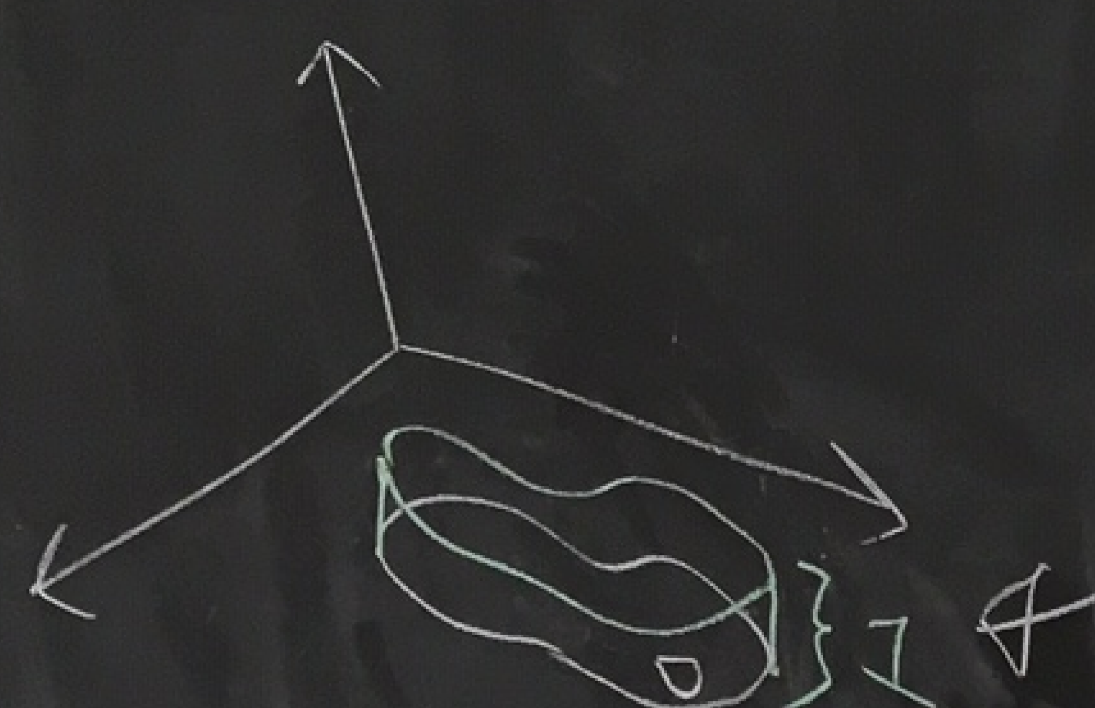
where c is a constant

- $D = D_1 \cup D_2$ and D_1 and D_2 don't overlap.



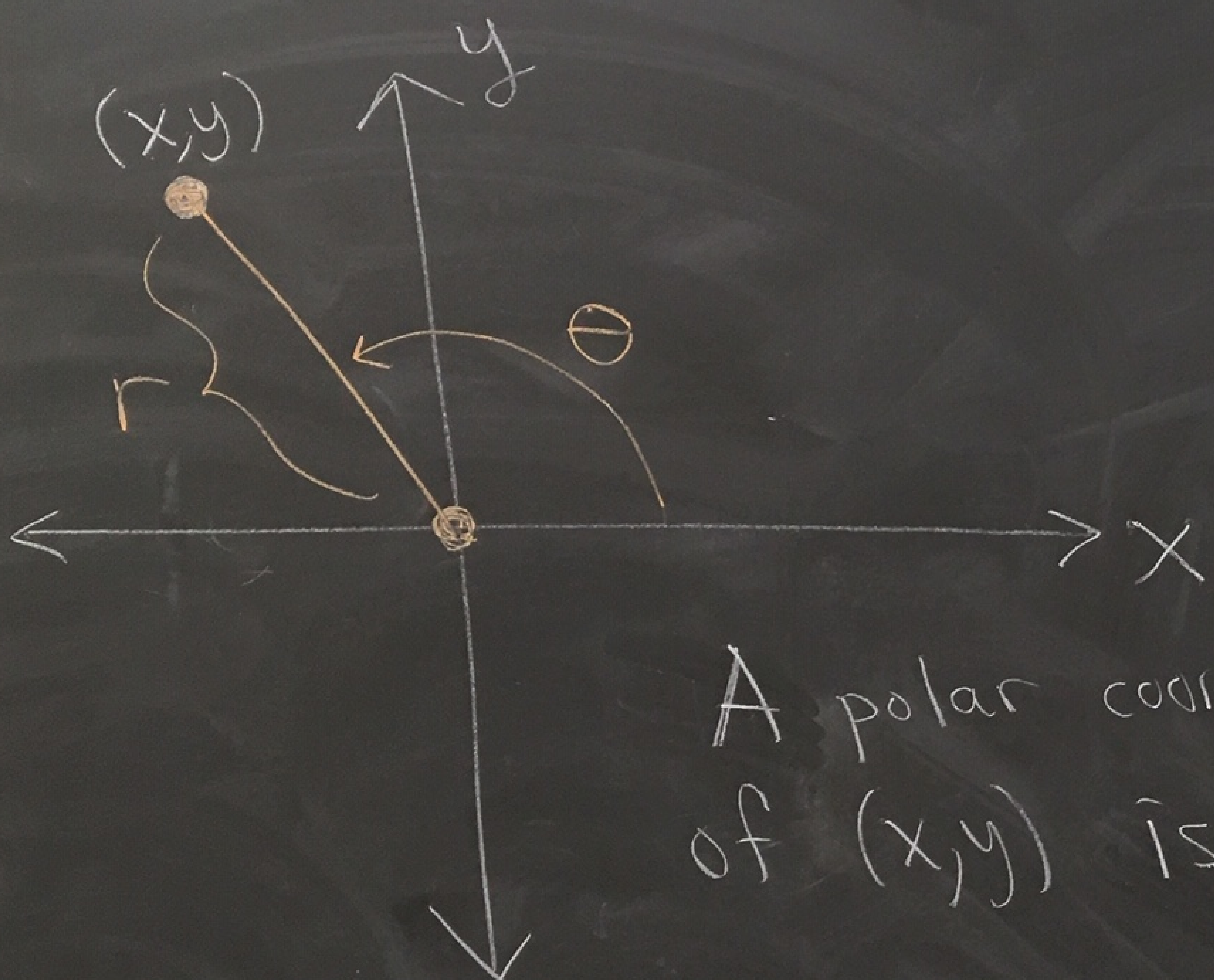
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

- $\iint_D 1\,dA$ = area of D.



volume of this solid $= \iint_D 1\,dA$

13.3 – Double integrals in Polar Coordinates



$$r^2 = x^2 + y^2$$
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

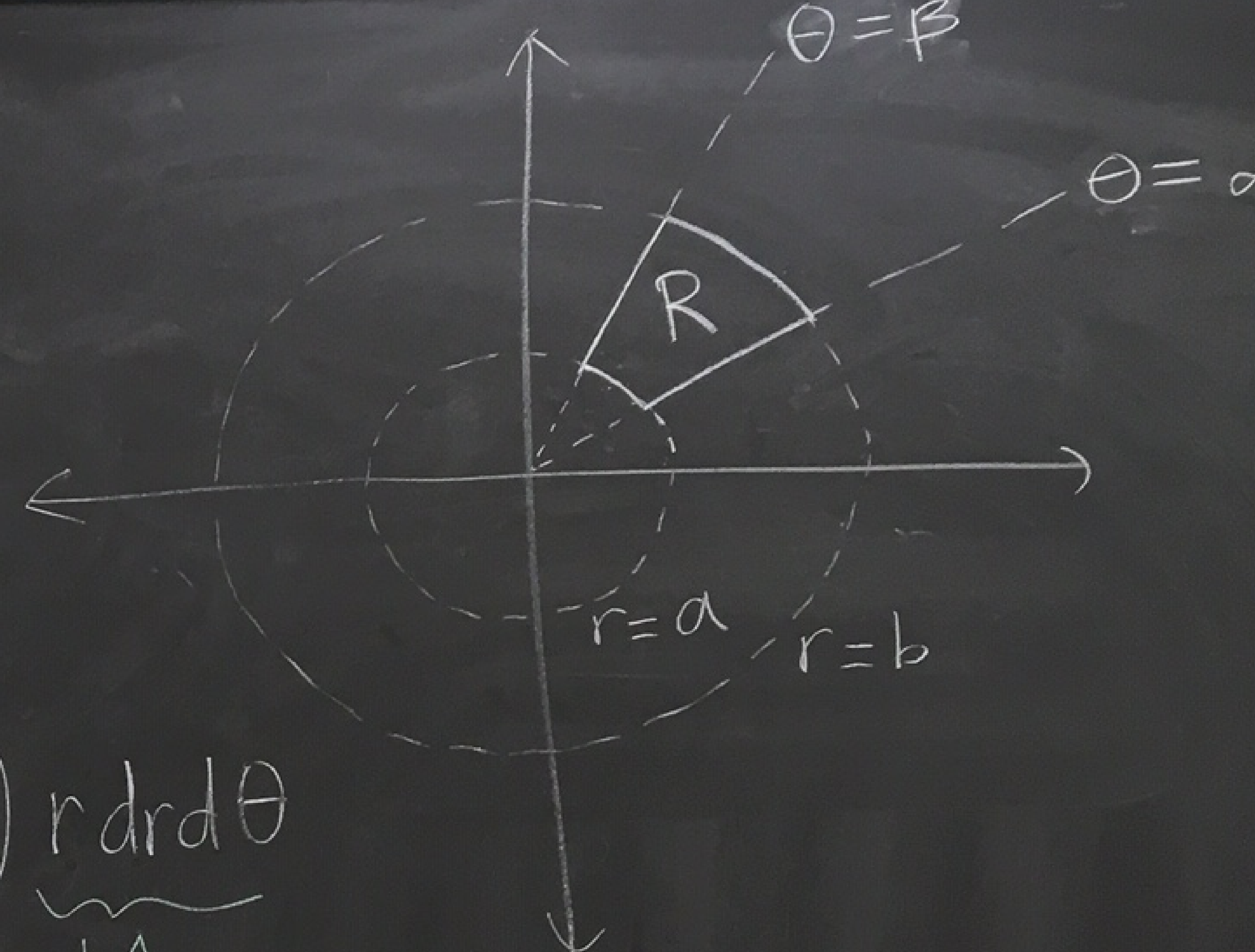
A polar coordinate of (x,y) is (r,θ).

If $f(x,y)$ is continuous on a polar rectangle R given by

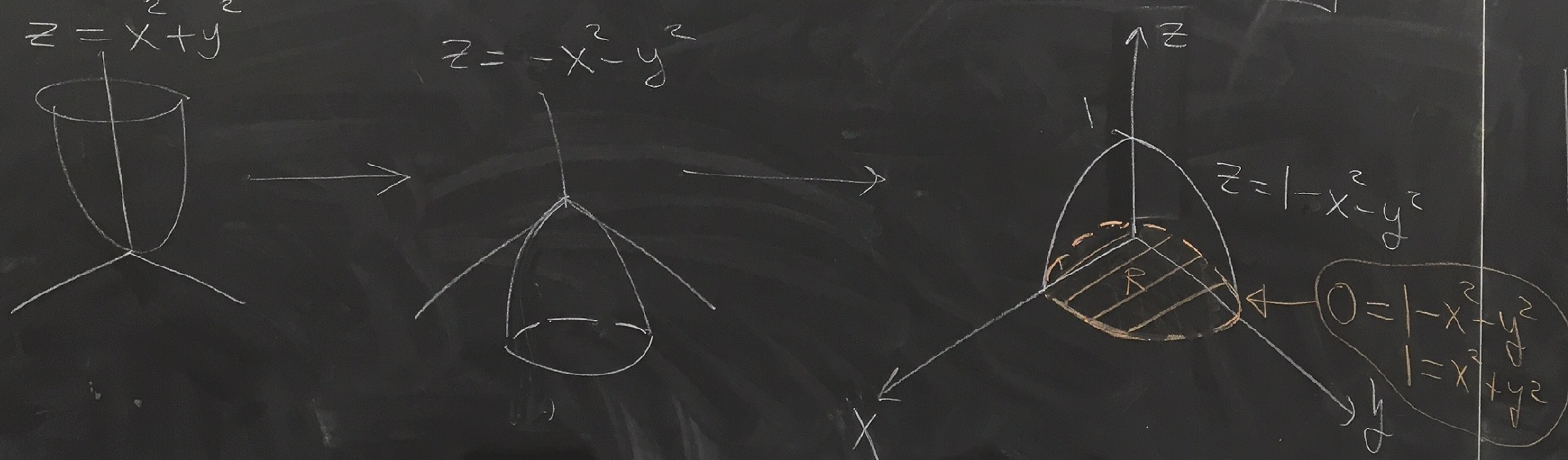
$$0 \leq a \leq r \leq b \quad \text{and} \quad \alpha \leq \theta \leq \beta$$

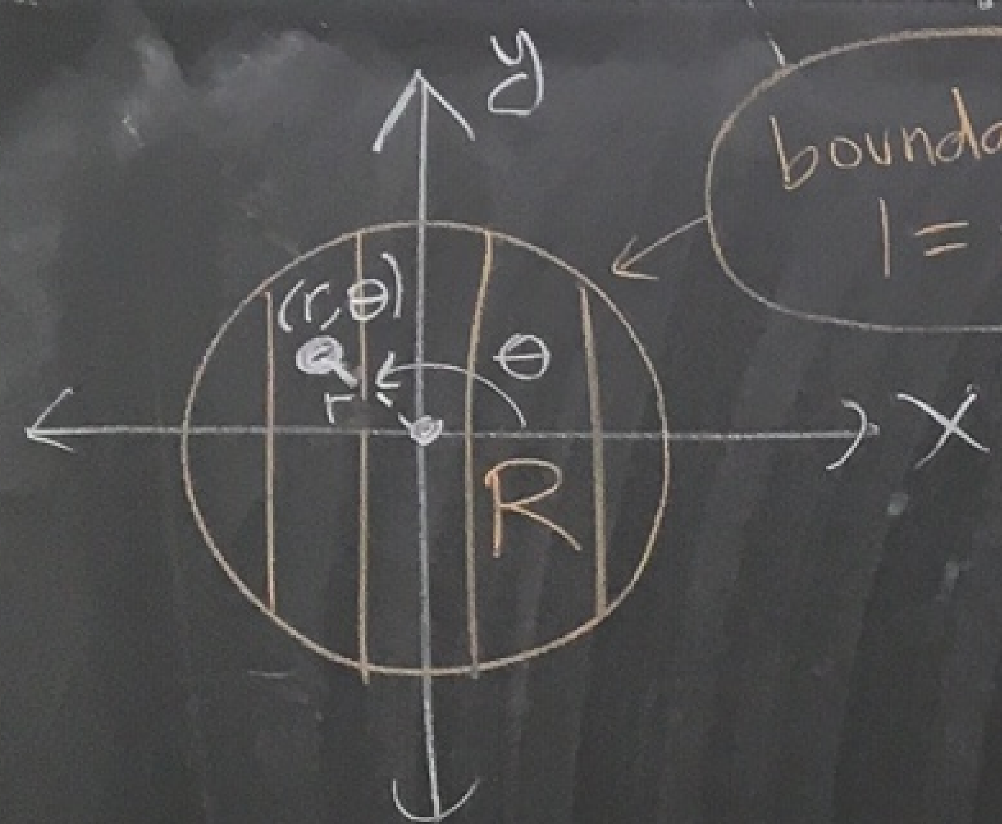
where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\,\underset{\sim}{dA} = \int_\alpha^\beta \int_a^b f(\underbrace{r\cos(\theta)}_{x}, \underbrace{r\sin(\theta)}_{y})\,\underbrace{r\,dr\,d\theta}_{dA}$$



Ex: Find the volume of the solid bounded by the plane $z = 0$ ← (xy-plane) and the paraboloid $z = 1 - x^2 - y^2$.





boundary is
$1 = x^2 + y^2$

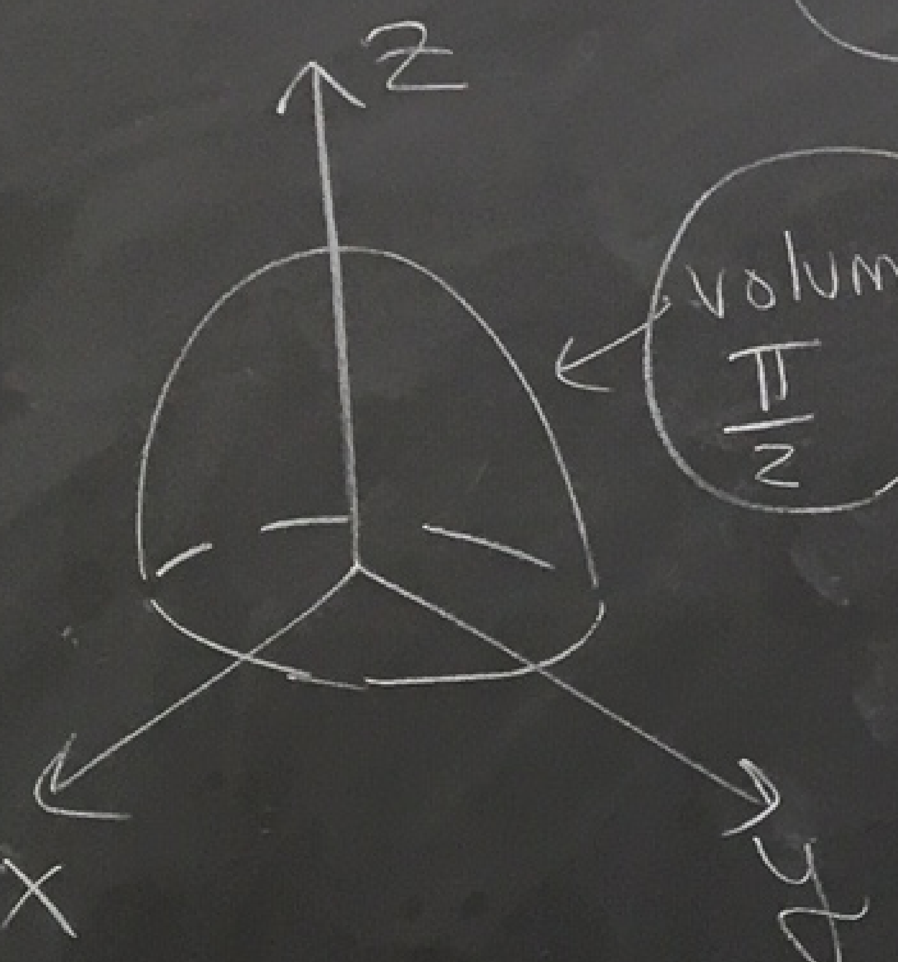
$$\boxed{\text{Volume of solid}} = \iint_R (1 - x^2 - y^2)\, dA$$

$$\boxed{\begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}}$$

$$= \int_0^{2\pi}\int_0^1 \Big(\underbrace{1 - (r\cos(\theta))^2 - (r\sin(\theta))^2}_{1 - x^2 - y^2}\Big)\, \underbrace{r\,dr\,d\theta}_{dA}$$

x^2+y^2

x^2+y^2



volume $\frac{\pi}{2}$

$$= \int_0^{2\pi}\int_0^1 \left(\underbrace{1 - \overbrace{r^2\cos^2(\theta) - r^2\sin^2(\theta)}^{1 - r^2(\cos^2(\theta)+\sin^2(\theta))}}_{(1-r^2)}\right) r\,dr\,d\theta$$

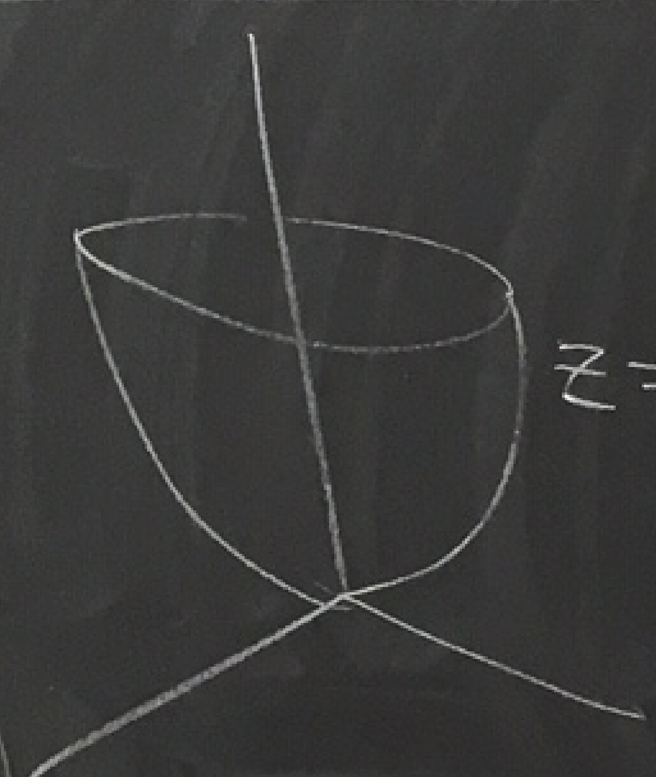
$$(1 - r^2) * r$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{1} d\theta$$

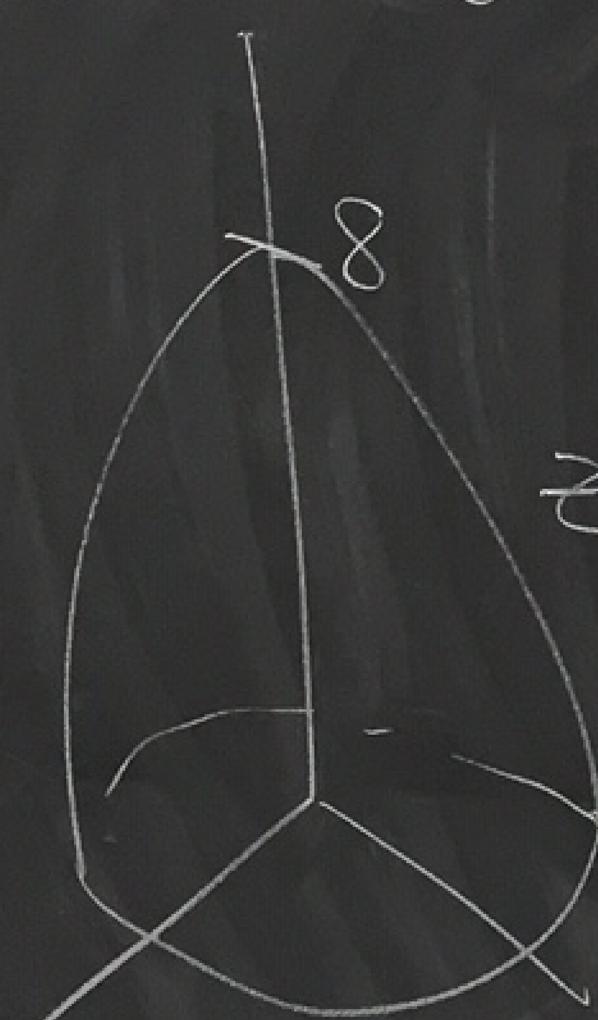
$$= \int_0^{2\pi}\left(\left(\frac{1}{2} - \frac{1}{4}\right) - (0)\right) d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{2\pi} = \frac{1}{4}(2\pi)$$

$$= \boxed{\frac{\pi}{2}}$$

<u>Ex</u>: Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$



$z = x^2 + y^2$



$z = 8 - x^2 - y^2$

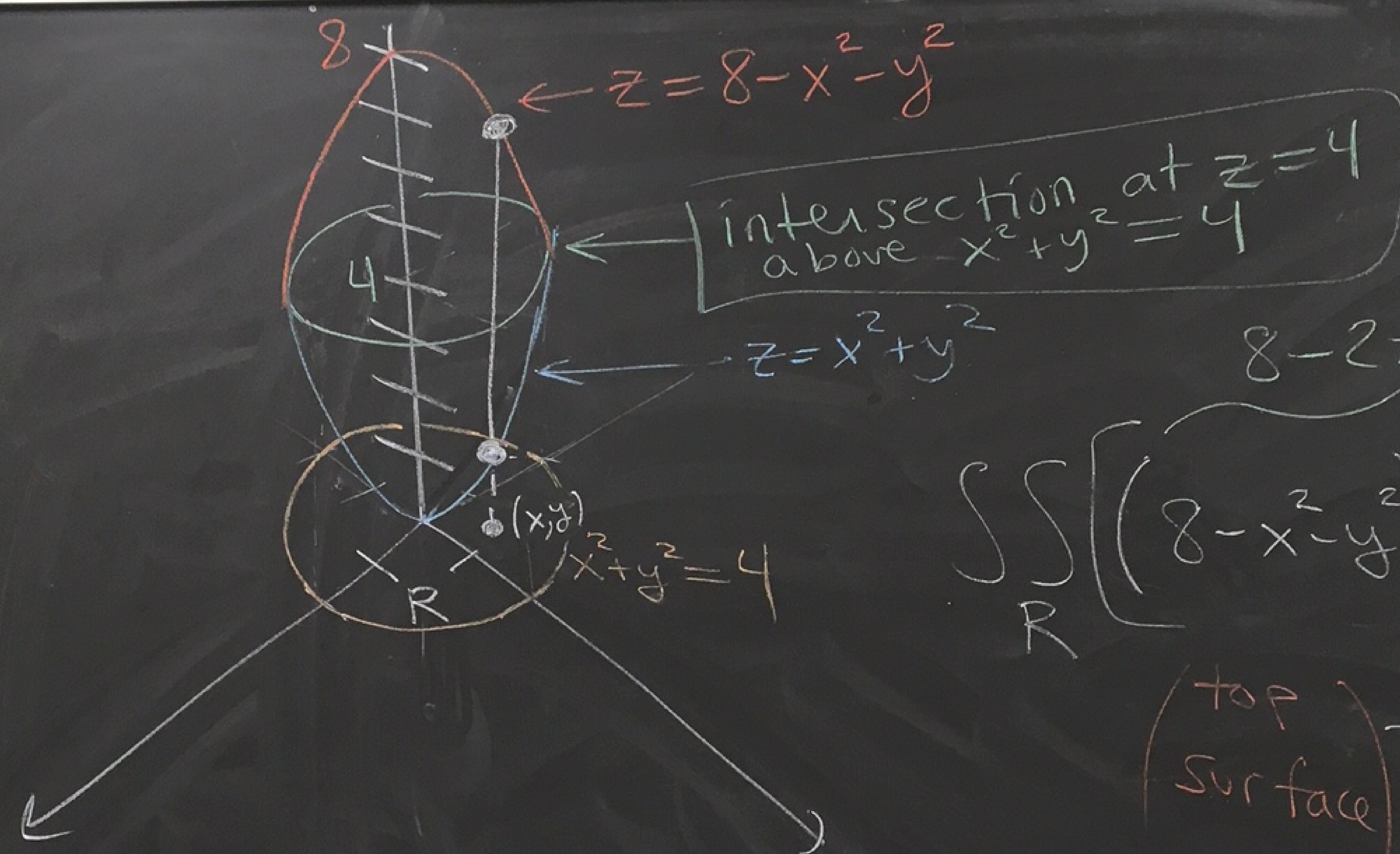
Where do the surfaces intersect?

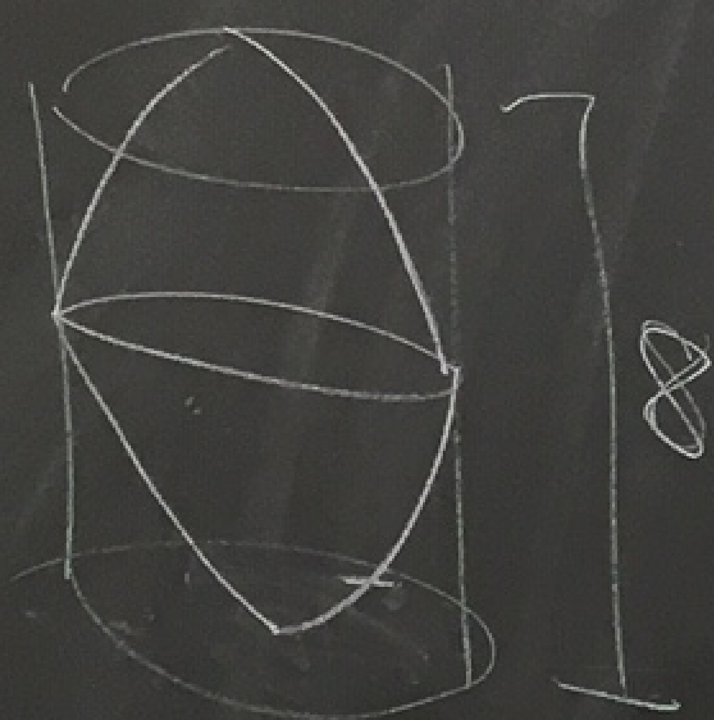
$$\underline{x^2 + y^2} = z = \underline{8 - x^2 - y^2}$$

$$2x^2 + 2y^2 = 8$$

$$\boxed{x^2 + y^2 = 4} \leftarrow \text{intersection}$$

So, $z = 4$ at the intersection

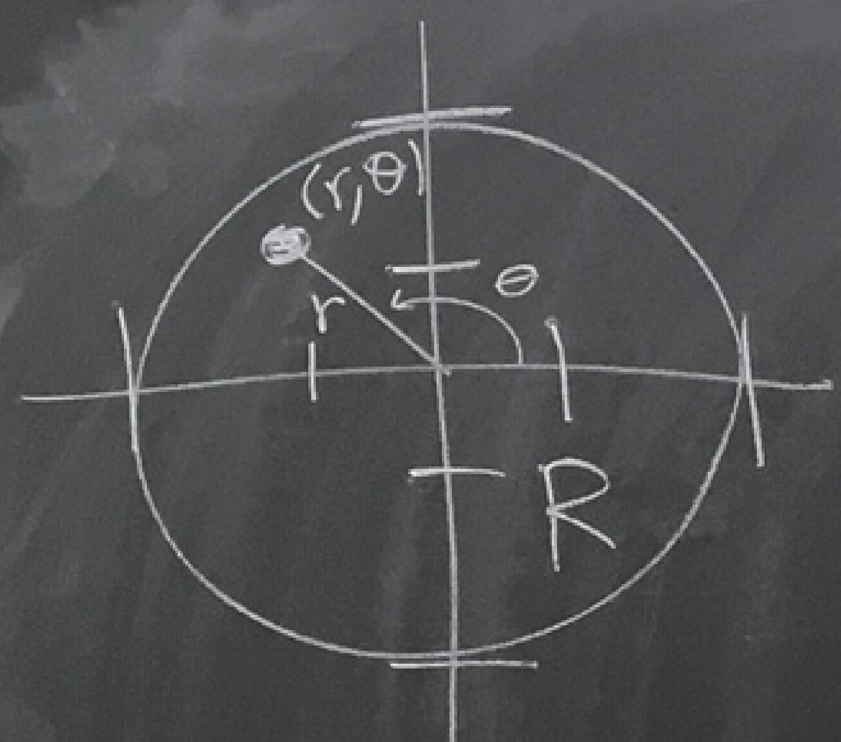




cylinder has volume $8 \cdot 4\pi = 32\pi$

$$\iint_R \left[\overbrace{(8-x^2-y^2)-(x^2+y^2)}^{8-2x^2-2y^2} \right] dA$$

$$\begin{pmatrix}\text{top}\\ \text{surface}\end{pmatrix} - \begin{pmatrix}\text{bottom}\\ \text{surface}\end{pmatrix}$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_0^{2\pi}\int_0^2 \underbrace{(8-2r^2)}_{8-x^2-y^2}\ \underbrace{r\,dr\,d\theta}_{dA}$$

$$8-2x^2-y^2$$
$$= 8-2(x^2+y^2)$$
$$= 8-2r^2$$

or you could plug in

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$$= \int_0^{2\pi}\int_0^2 (8r-2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \Big(\underbrace{8\frac{r^2}{2} - 2\frac{r^4}{4}}_{4r^2-\frac{r^4}{2}}\Big)\Big|_0^2\,d\theta = \int_0^{2\pi}\Big(\Big(4\cdot 2^2 - \frac{2^4}{2}\Big) - 0\Big)\,d\theta$$

$$= \int_0^{2\pi} 8\,d\theta = 8\theta\Big|_0^{2\pi} = 8(2\pi)$$

$$= 16\pi$$