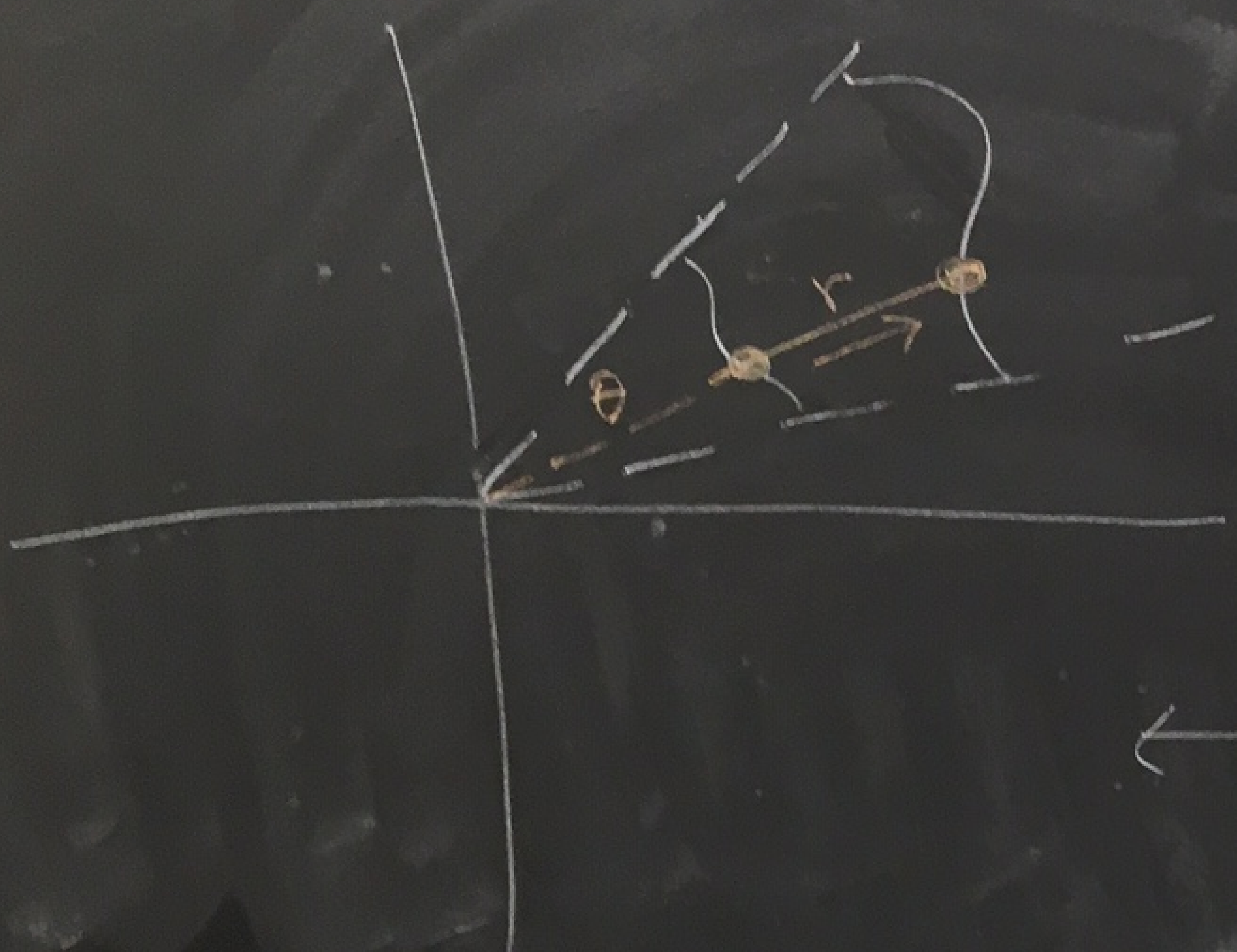
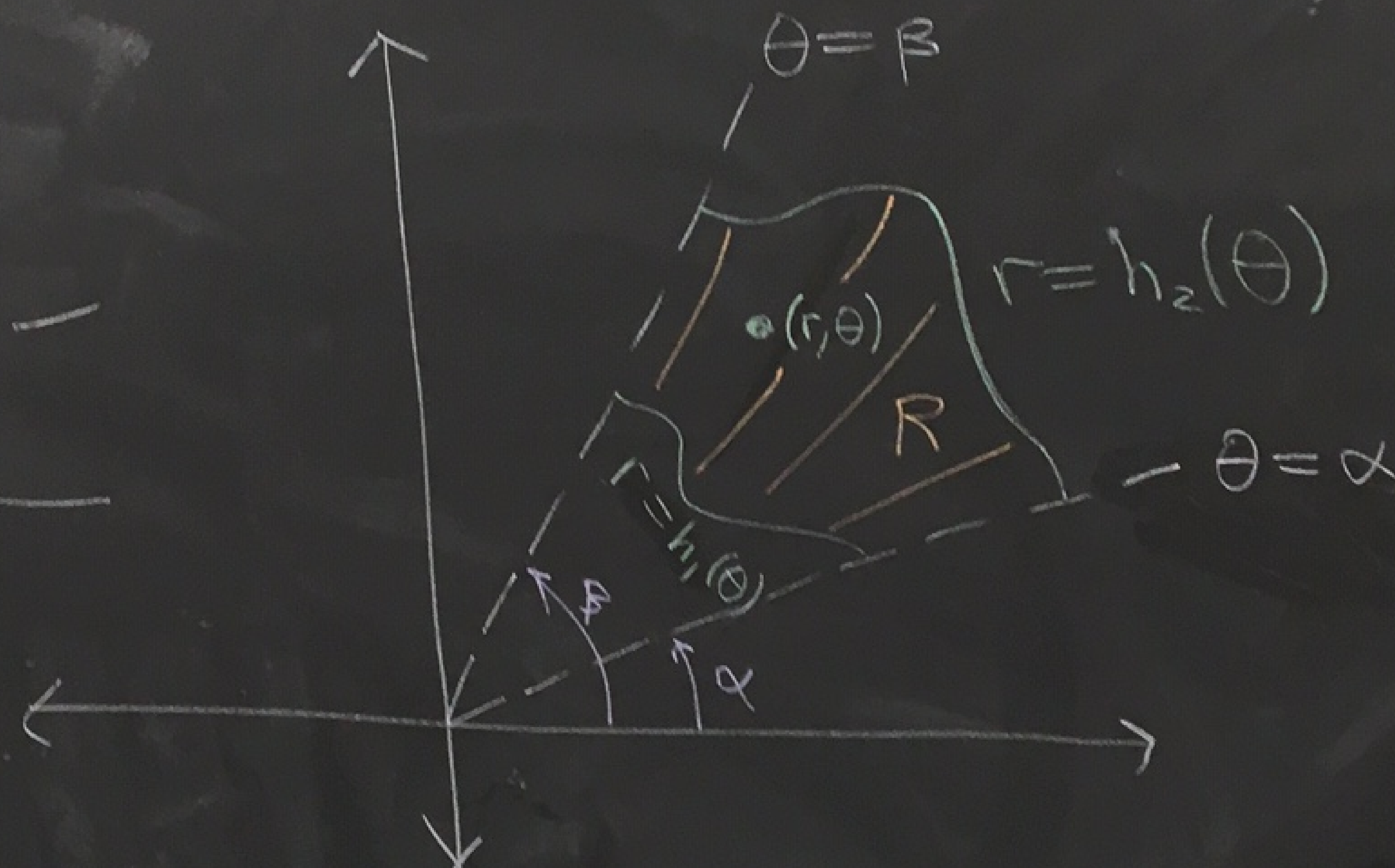
Thursday
10/24

(13.3 continued...)





Suppose f is continuous on a region R consisting of all points (r,θ) satisfying

$$h_1(\theta) \le r \le h_2(\theta)$$

$$\alpha \le \theta \le \beta$$

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where $0 \le \beta - \alpha \le 2\pi$, then

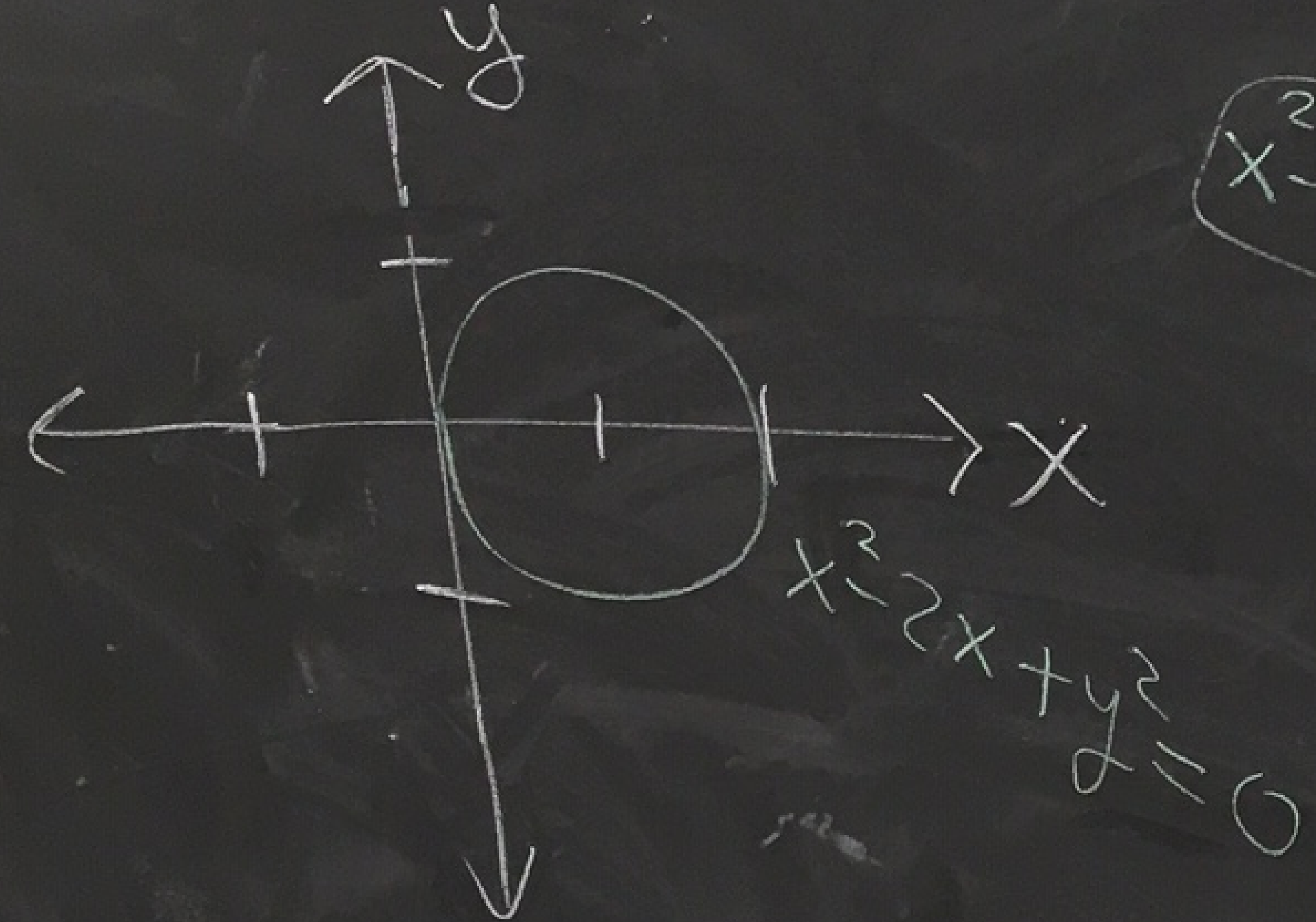
$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta}$$

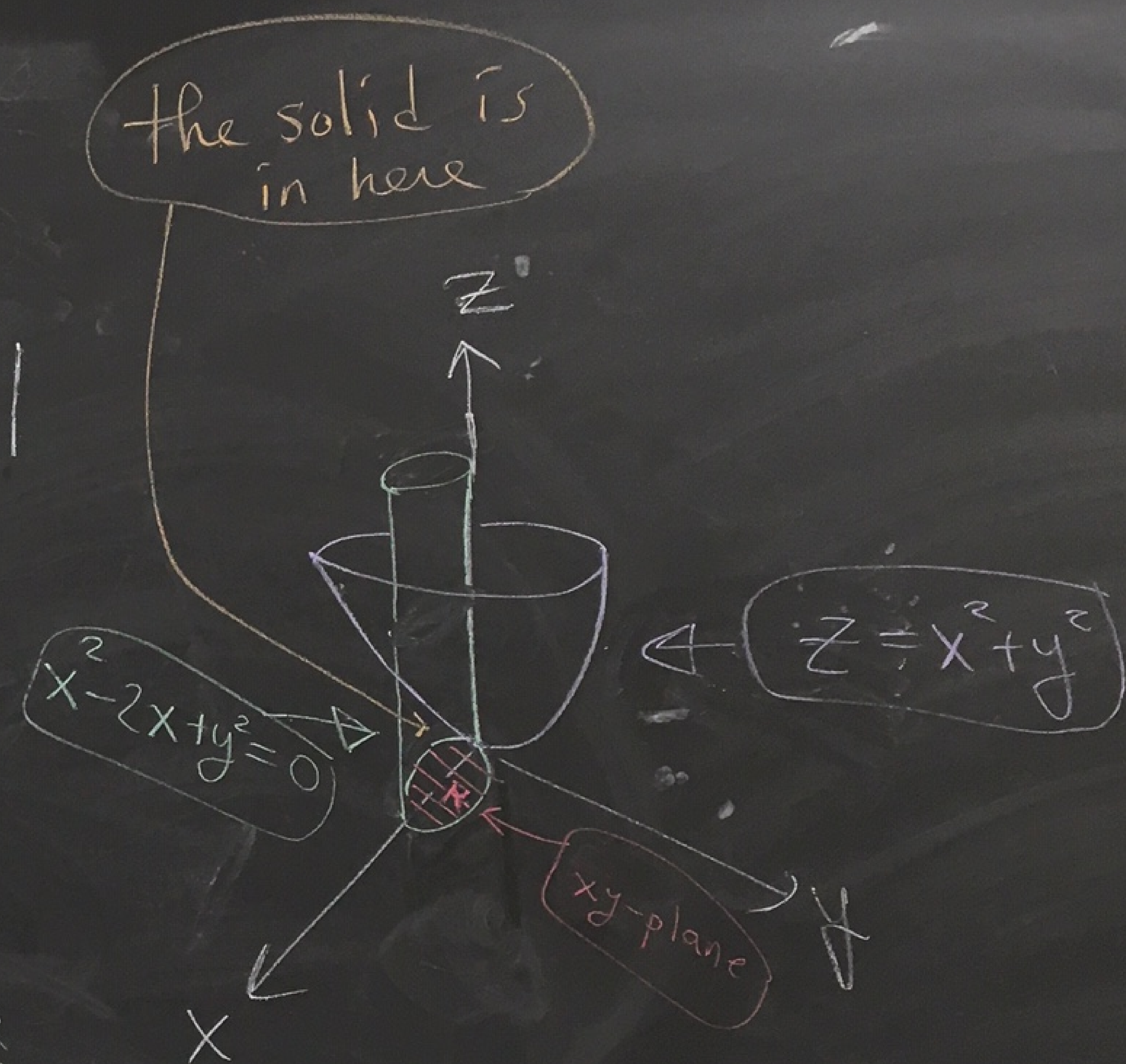
Ex: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

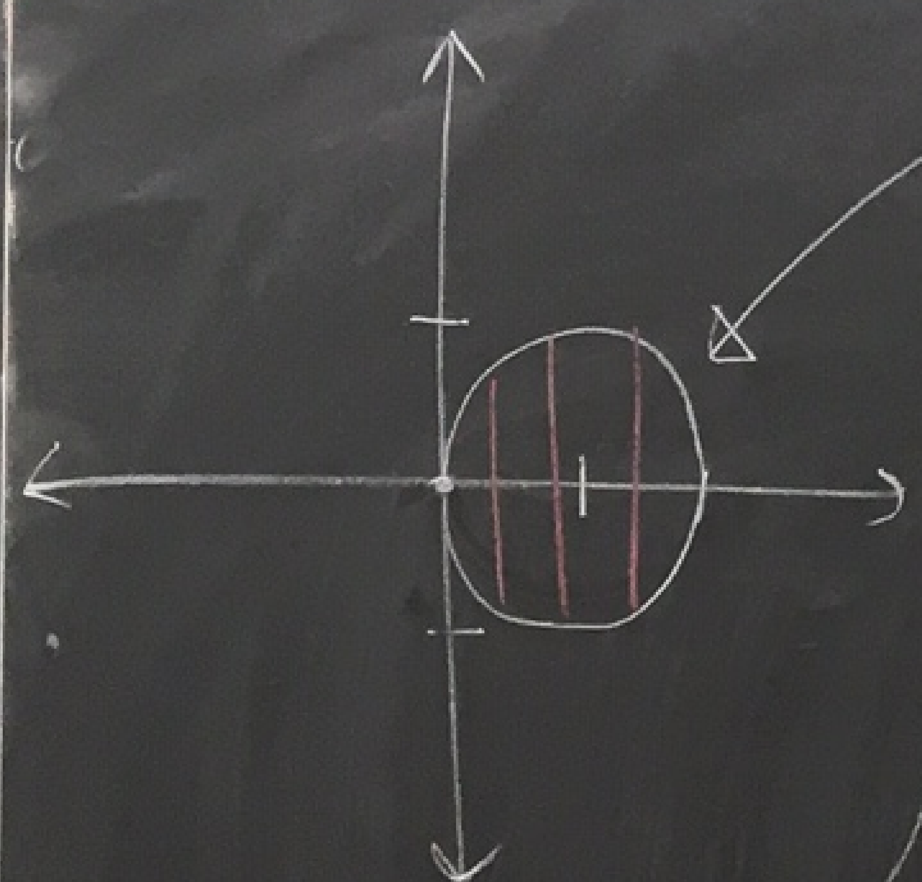
$$x^2-2x+y^2=0$$

$$x^2-2x+1+y^2=1$$

$$(x-1)^2+(y-0)^2=1$$







$$x^2 - 2x + y^2 = 0$$

$$(x^2 + y^2) - 2x = 0$$

$$r^2 - 2r\cos(\theta) = 0$$

$$r - 2\cos(\theta) = 0$$

$$\boxed{r = 2\cos(\theta)}$$

Change to polar

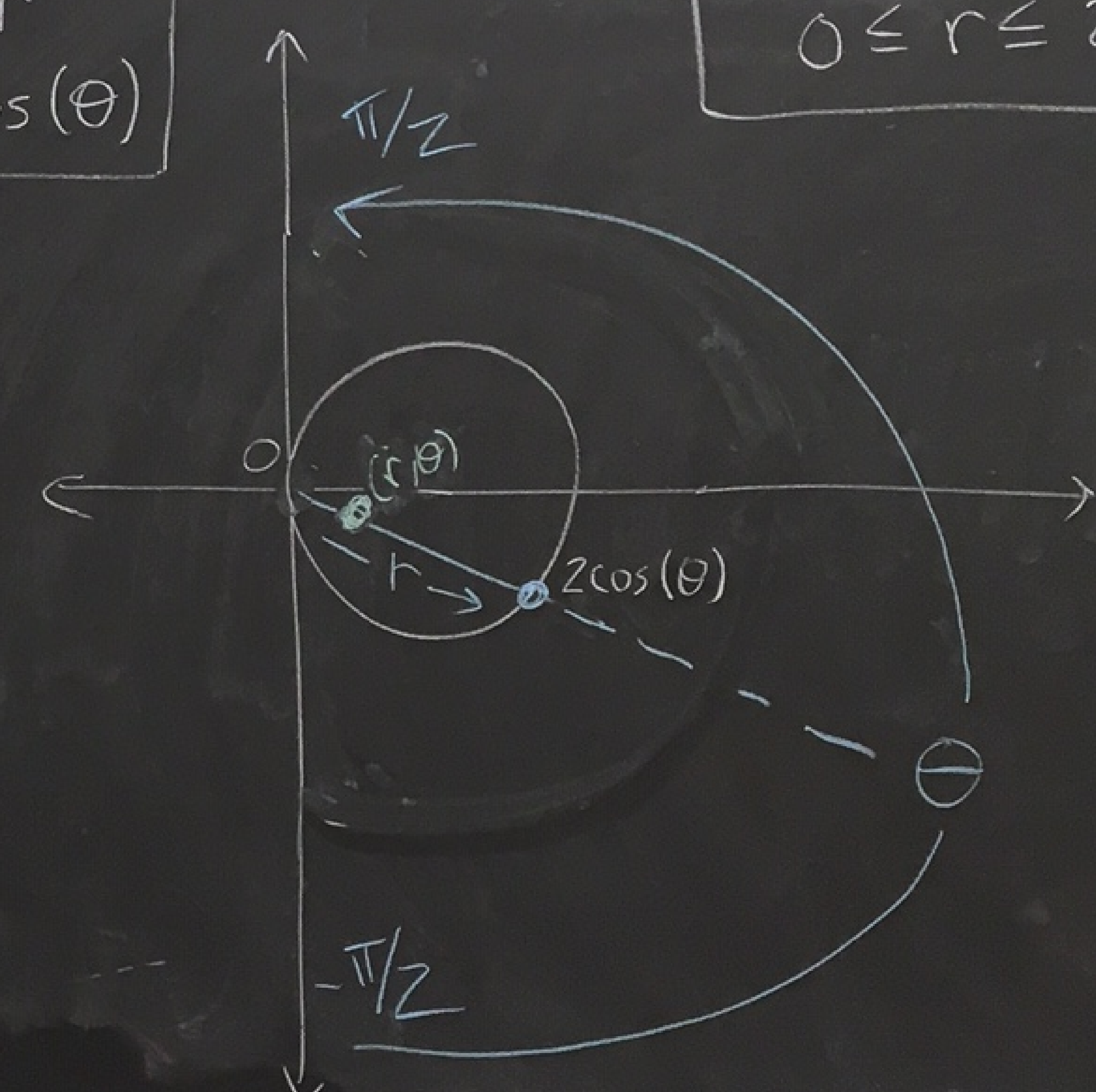
$$x^2 + y^2 = r^2$$

$$x = r\cos(\theta)$$

Was this ok to divide by r?

$r = 0$ gives the point $(0,0)$

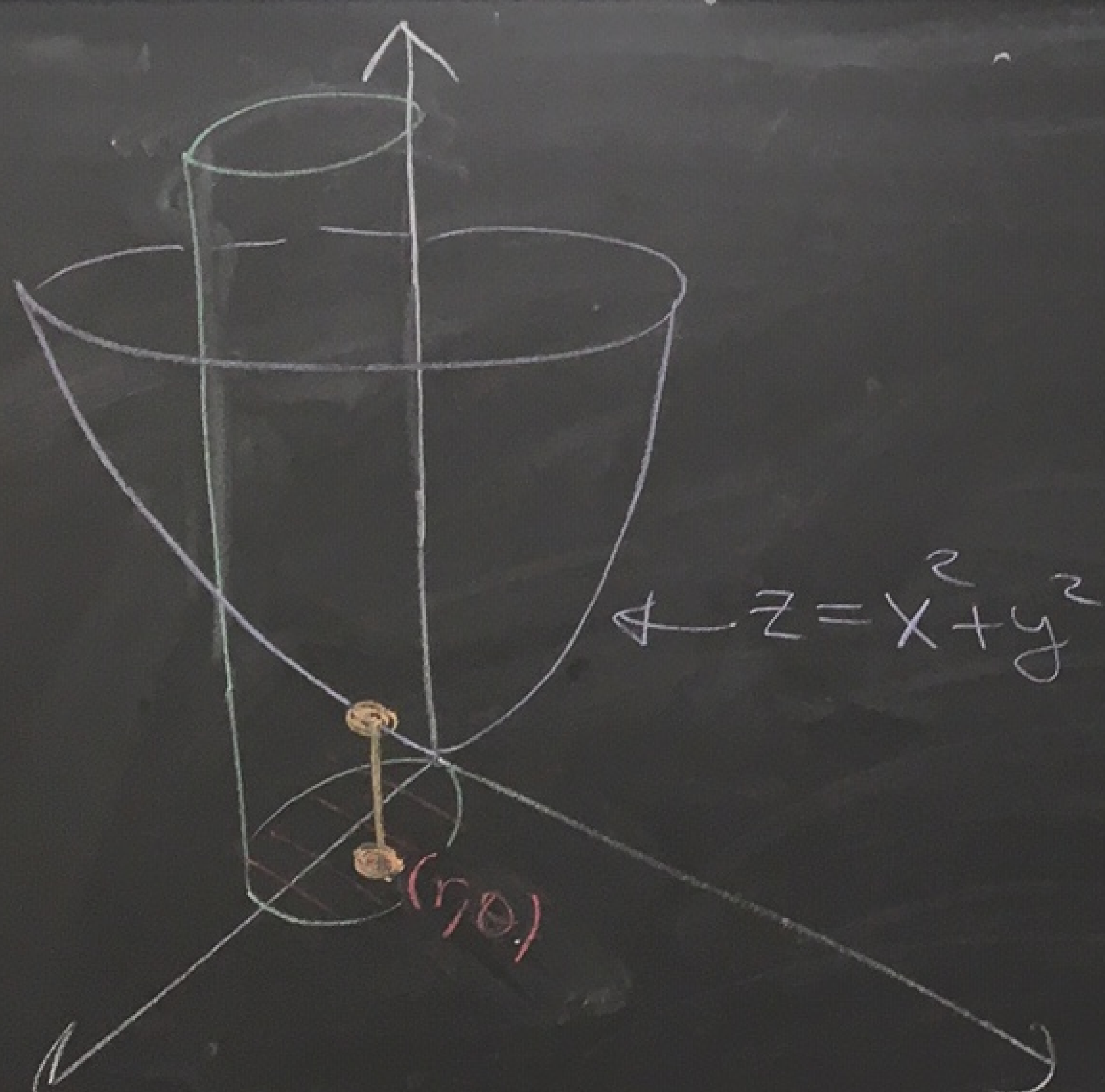
If $-\frac{\pi}{2} \le \theta \le \pi/2$ then you start and end at $(0,0)$. So we didn't lose $(0,0)$ when we divided by r



Parameterization of R

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 2\cos(\theta)$$

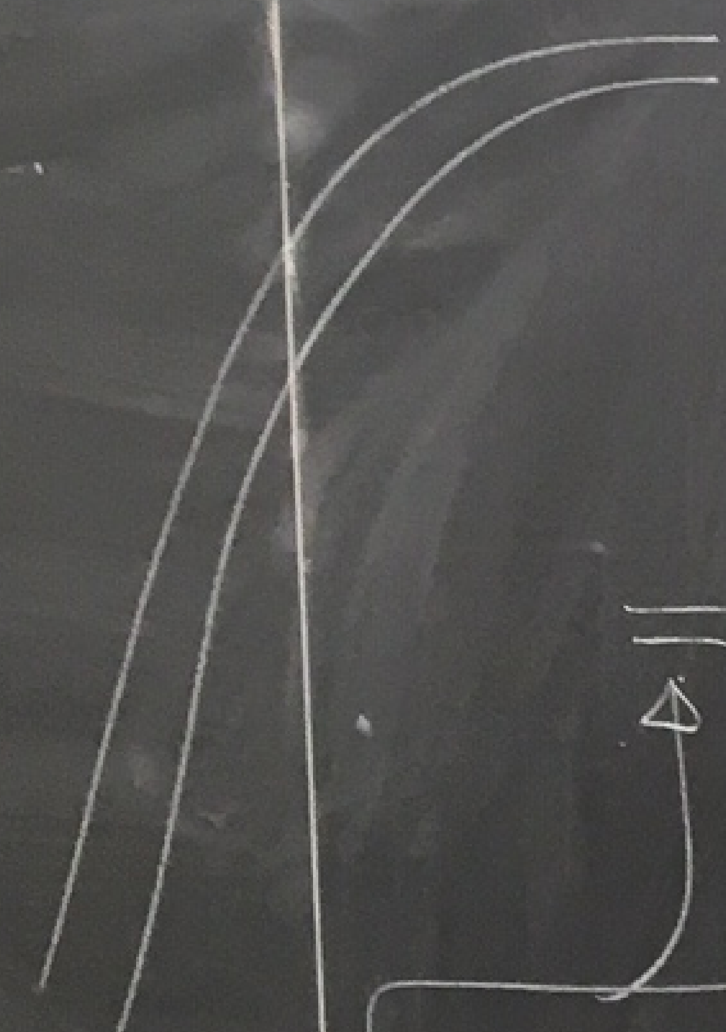


$$V = \iint_R (x^2+y^2)\,dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^2 \cdot r\,dr\,d\theta$$

$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ x^2+y^2 &= r^2 \\ dA &= r\,dr\,d\theta \end{aligned}}$$

$$= \int_{-\pi/2}^{\pi/2} \left. \frac{r^4}{4} \right|_{r=0}^{2\cos(\theta)} d\theta$$



$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 2^4\cos^4(\theta)\,d\theta = 4\int_{-\pi/2}^{\pi/2}\left(\cos^2(\theta)\right)^2 d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2}\left(\frac{1}{2}\right)^2\left(1+2\cos(2\theta)+\cos^2(2\theta)\right)d\theta = \int_{-\pi/2}^{\pi/2}\left(1+2\cos(2\theta)+\frac{1}{2}+\frac{1}{2}\cos(4\theta)\right)d\theta$$

$$\boxed{\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} = \frac{1}{2}\left(1+\cos(2\theta)\right)}$$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta)\right)d\theta$$

$$= \frac{3}{2}\theta + 2\cdot\frac{1}{2}\sin(2\theta) + \frac{1}{2}\cdot\frac{1}{4}\sin(4\theta)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{3}{2}\left(\frac{\pi}{2}\right) + \underbrace{\sin(\pi)}_{0} + \frac{1}{8}\underbrace{\sin(2\pi)}_{0} \right) - \left(\frac{3}{2}\left(-\frac{\pi}{2}\right) + \underbrace{\sin(-\pi)}_{0} + \frac{1}{8}\underbrace{\sin(-2\pi)}_{0} \right)$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2} \approx 4.7123.$$

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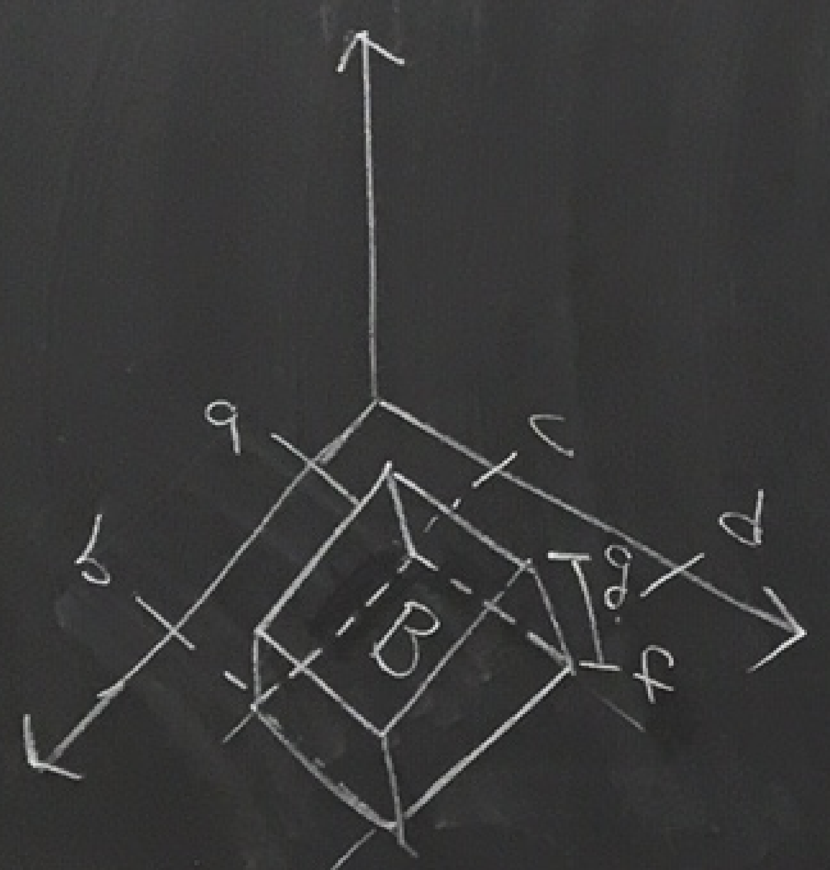
exists,

13.4 – Triple Integrals

Suppose that B is the box of all (x,y,z) with $a \le x \le b$ and $c \le y \le d$ and $f \le z \le g$.

Suppose $f(x,y,z)$ is a function of three variables.

We want to integrate f over B.



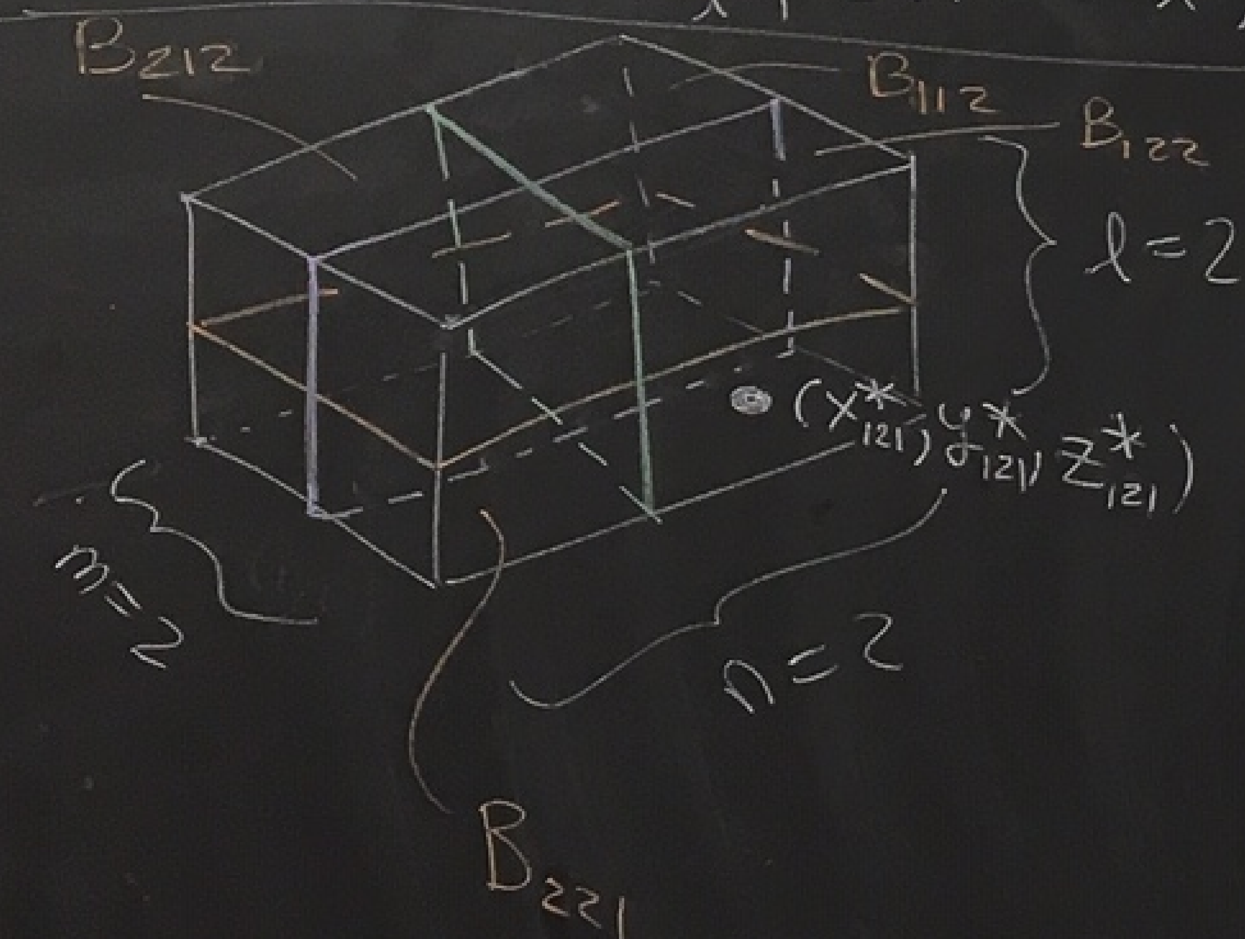
Break B into sub-boxes as follows:

- Divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of width $\Delta x = \frac{b-a}{n}$
- Divide $[c,d]$ into m subintervals $[y_{j-1}, y_j]$ of width $\Delta y = \frac{d-c}{m}$
- Divide $[f,g]$ into ℓ subintervals $[z_{k-1}, z_k]$ of width $\Delta z = \frac{g-f}{\ell}$

Let B_{ijk} be the subbox with (x, y, z) in the ranges

$$x_{i-1} \le x \le x_i, \quad y_{j-1} \le y \le y_j, \quad z_{k-1} \le z \le z_k.$$

$n = 2$
$m = 2$
$\ell = 2$
example



In each subbox B_{ijk} pick a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.

The integral of f over B, if it exists, is defined as

$$\iiint_B f(x,y,z)\,dV = \lim_{n,m,\ell \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{\ell} \underbrace{f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})}_{\text{function value at } (x^*_{ijk}, y^*_{ijk}, z^*_{ijk})} \underbrace{\Delta x \Delta y \Delta z}_{\text{volume of subboxes}}$$

13.4 – Triple Integ

se that B
$a \le x \le b$ and c