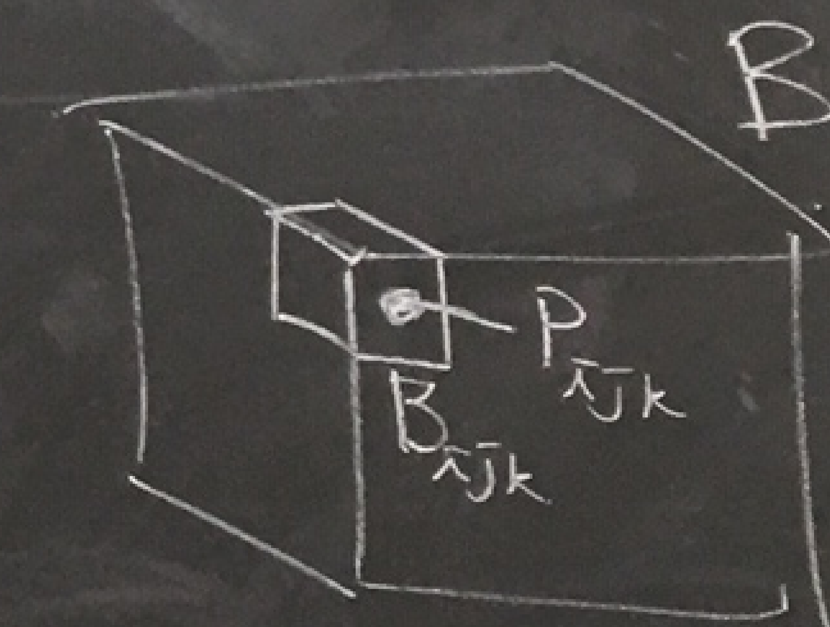
Tuesday
10/29

Test 2
11/14 – Thursday

12.8
12.9
13.1
13.2
13.3

Break B into little boxes. B_{ijk}



$$\iiint_B f\,dV = \lim_{\substack{\text{little}\\ \text{boxes}\\ \text{go to}\\ \text{size}\\ 0}} \sum_{\substack{\text{add}\\ \text{over}\\ \text{little}\\ \text{boxes}}} f(P_{ijk}) \cdot \text{Volume(little box)}$$

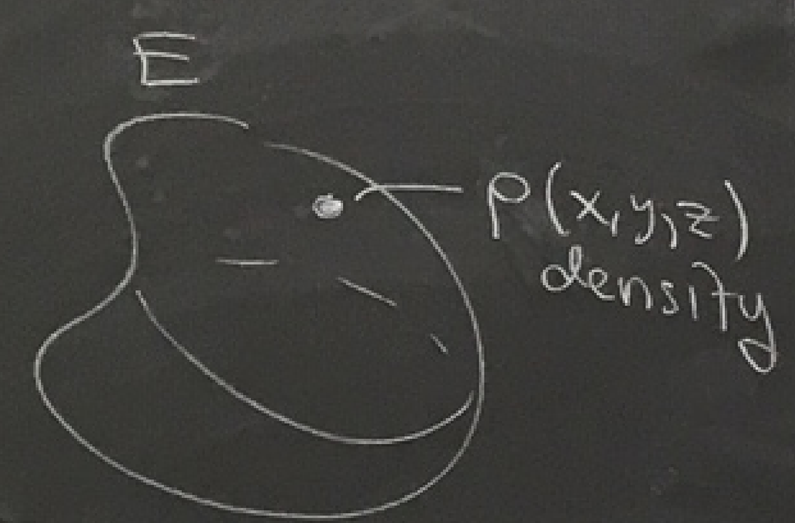
13

13.4 continued...

Fubini's Theorem

If $f(x,y,z)$ is continuous on the rectangular box $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ f \le z \le g\}$ then

$$\iiint_B f(x,y,z)\,dV = \underbrace{\int_f^g \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}_{\text{You can rearrange the order of } dx\,dy\,dz \text{ in any way you want}}$$



$$\text{mass} = \iiint_E \rho(x,y,z)\,dV$$

Ex: Evaluate $\iiint_B xyz^2\,dV$

over the box

$$B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

$$\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 (xyz^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_{-1}^2 \left[yz^2 \frac{x^2}{2} \right]_{x=0}^{1} dy\,dz =$$

$$\int_0^3 \int_{-1}^2 \frac{1}{2} y z^2 \, dy \, dz$$

$$= \int_0^3 \left(\frac{1}{2} z^2 \frac{y^2}{2} \Big|_{y=-1}^{2} \right) dz$$

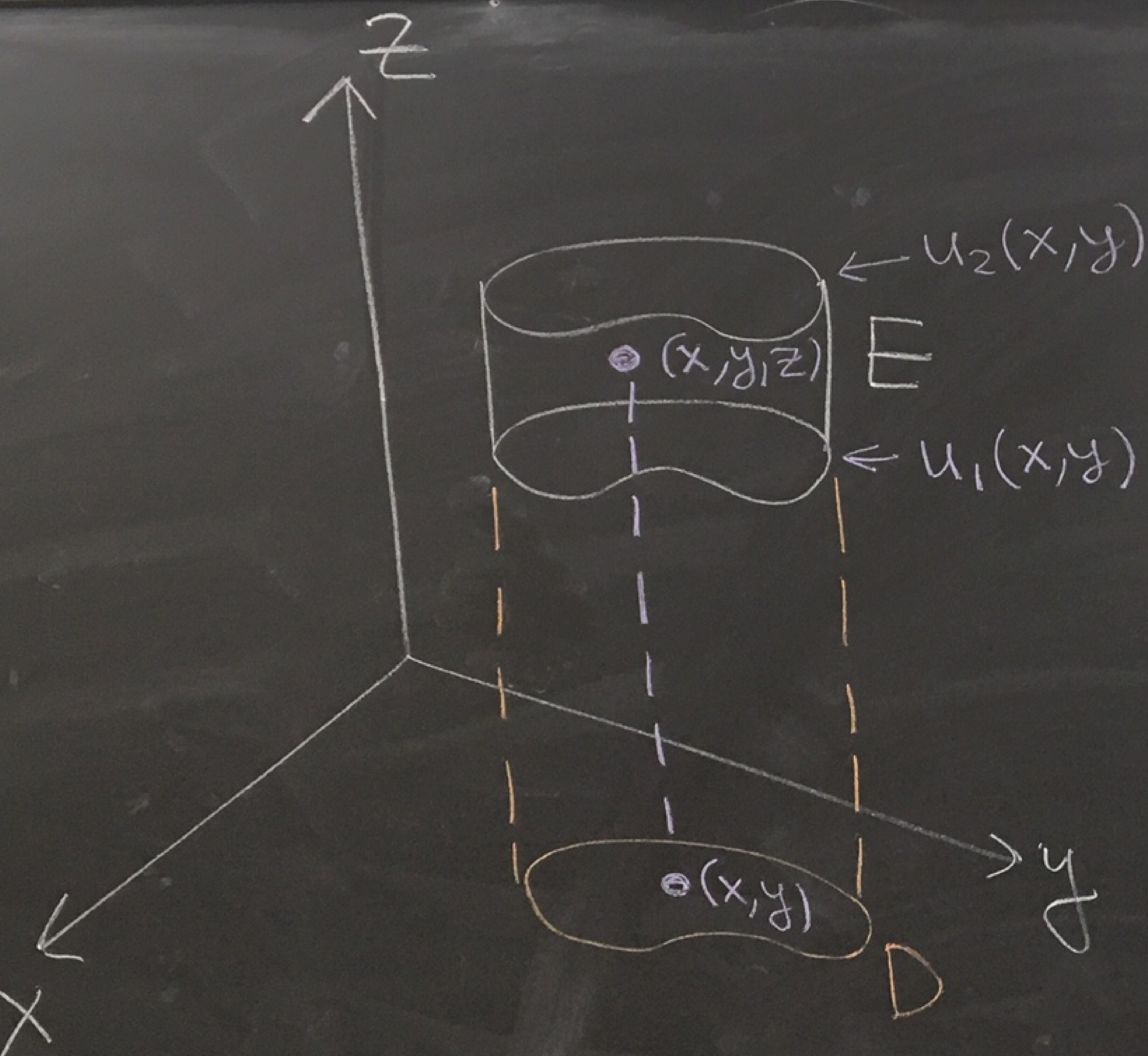
$$= \int_0^3 \frac{1}{4} z^2 \left(2^2 - (-1)^2 \right) dz$$

$$= \int_0^3 \frac{3}{4} z^2 \, dz$$

$$\frac{3}{4} \frac{z^3}{3} \Big|_0^3$$

$$= \frac{1}{4}(3^3) - \frac{1}{4}(0^3)$$

$$= \frac{27}{4}$$



Let E be a solid region. Let D be the projection of E into the xy-plane.

Suppose that E consists of all (x,y,z) where (x,y) is in D and $u_1(x,y) \leq z \leq u_2(x,y)$

If $f(x,y,z)$ is continuous on E then

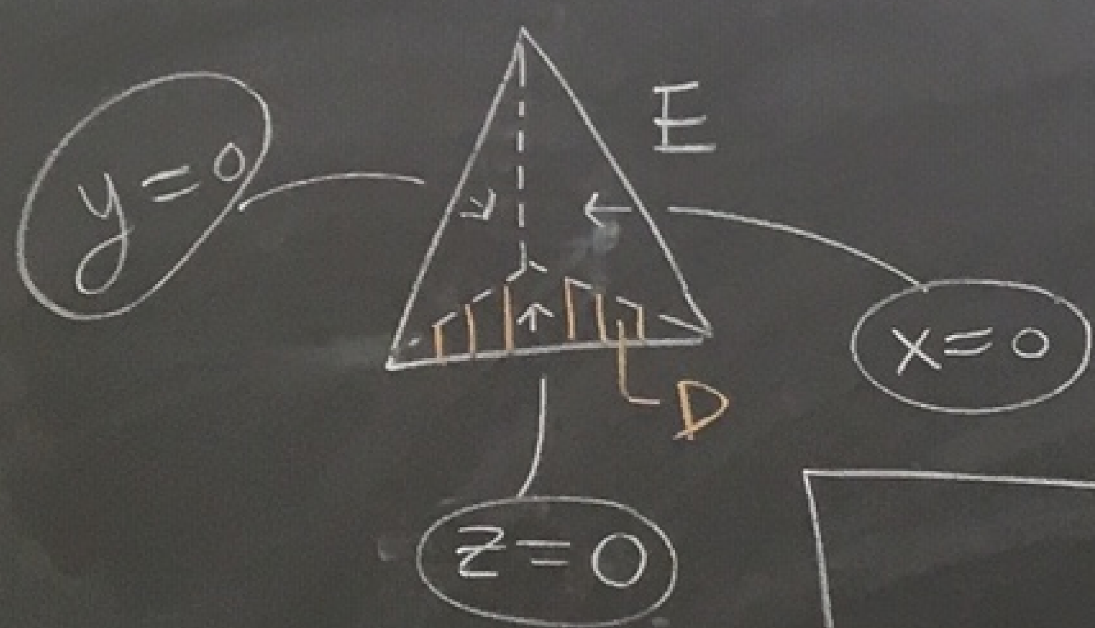
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

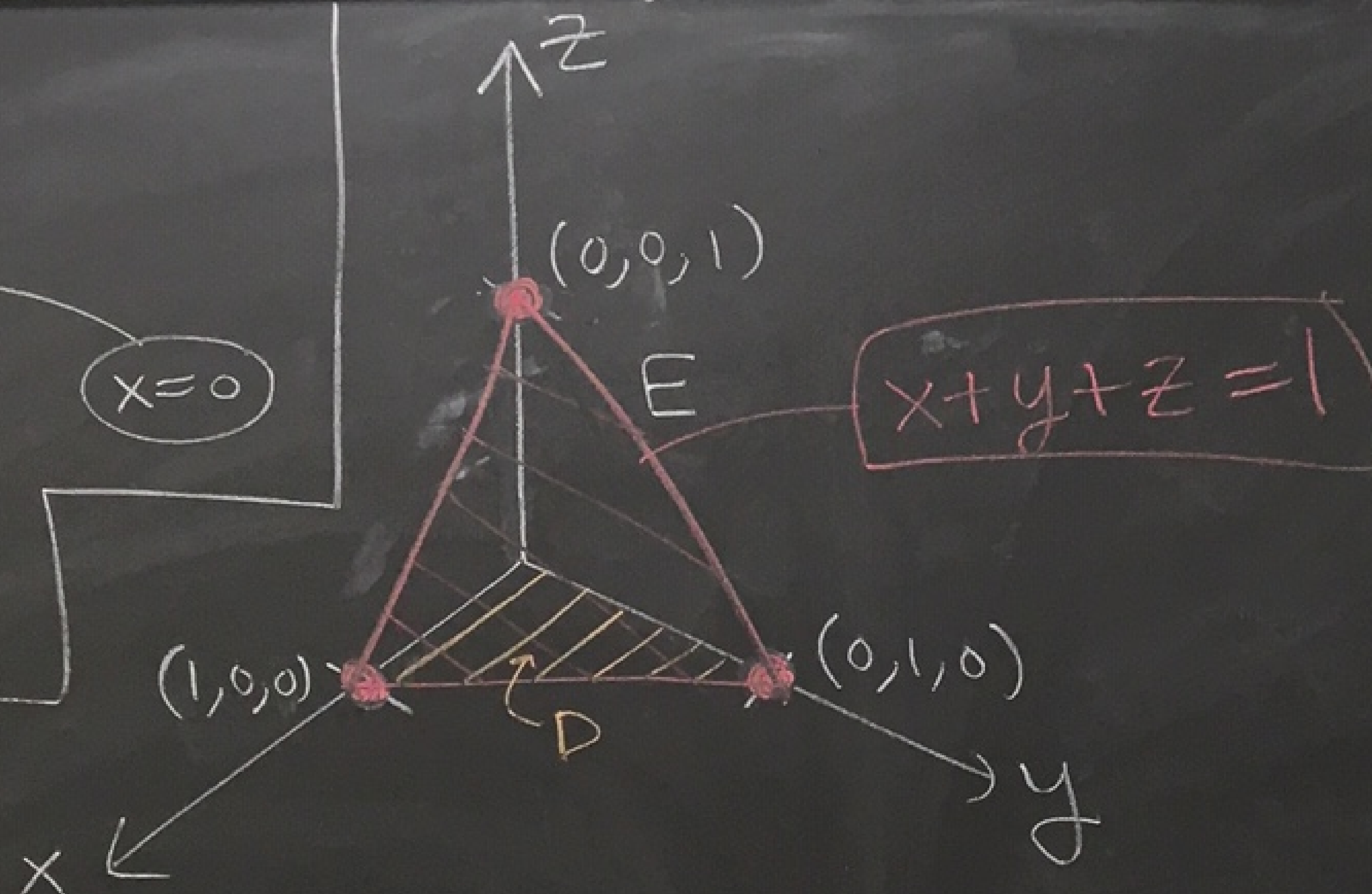
<u>Ex:</u> Evaluate $\iiint_E z\,dV$ where E is the solid <u>tetrahedron</u> bounded by the four planes

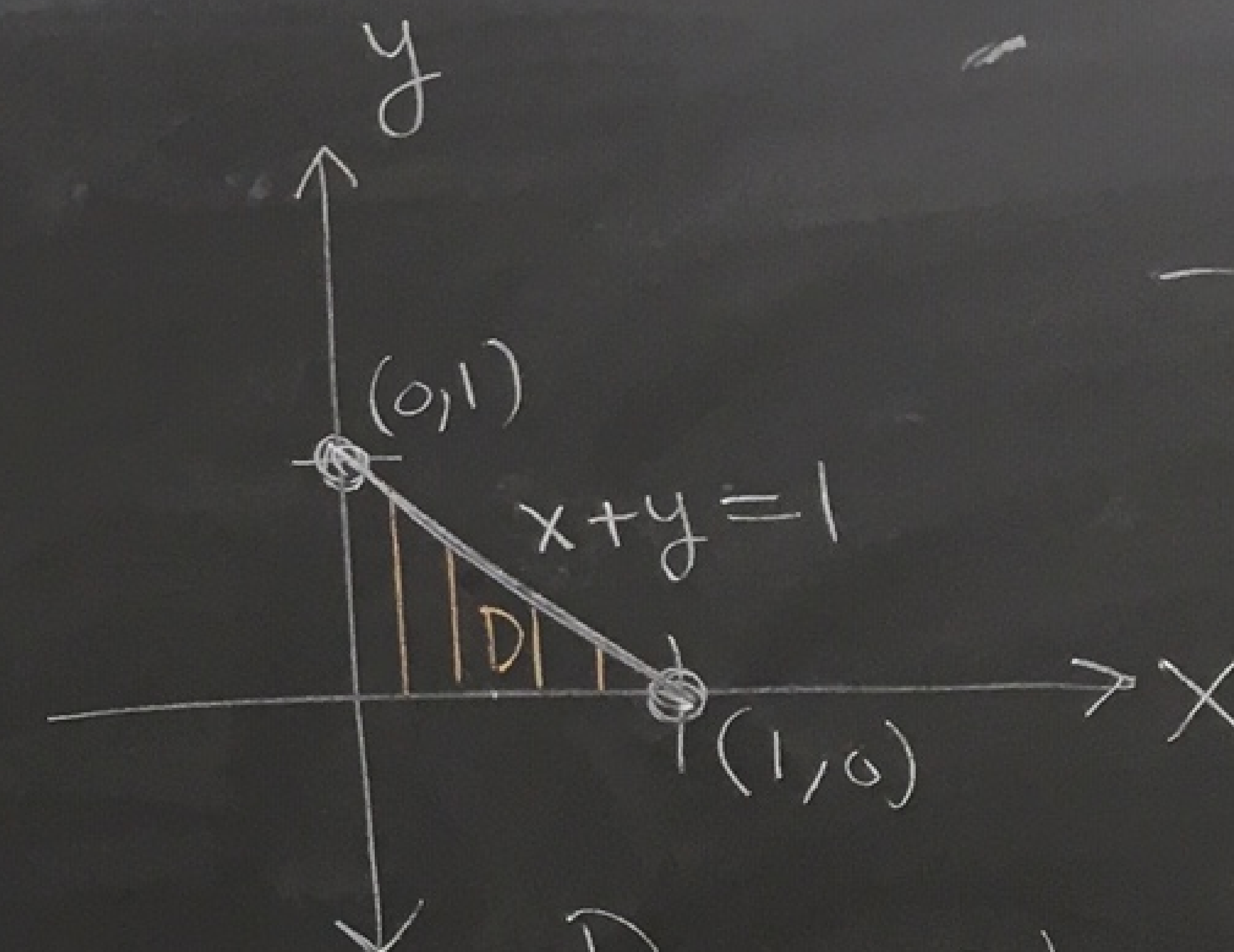
$x=0$, $y=0$, $z=0$, and $x+y+z=1$

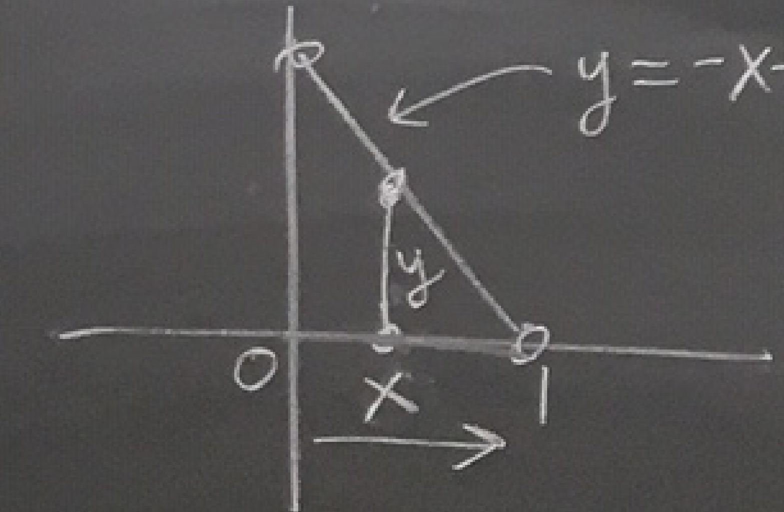
D

$_2(x,y)$









$(0,0,1), (0,1,0), (1,0,0)$ are all on $x+y+z=1$.

Since $x+y+z=1$ is a plane it contains the lines between these three points.

D consists of all (x,y) with

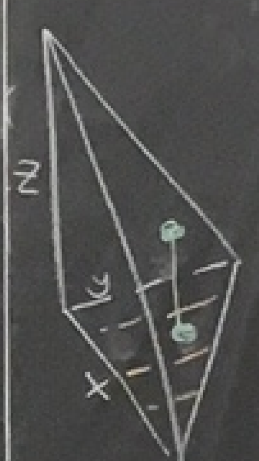
$$0 \le x \le 1$$

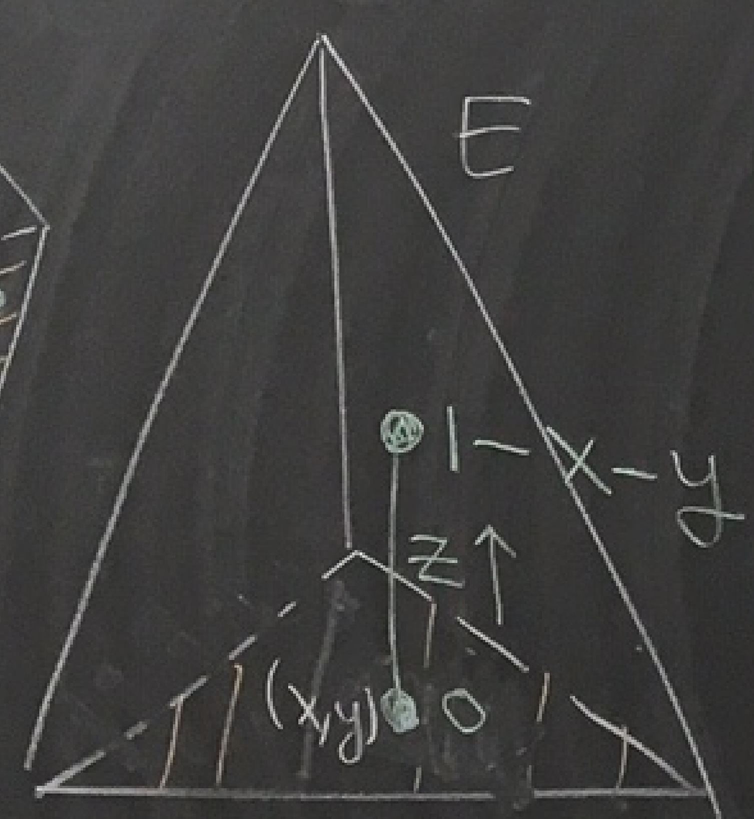
$$0 \le y \le -x+1$$

dx

dx

$$\iiint_E z\,dV = \underbrace{\int_0^1 \int_0^{-x+1}}_{D} \underbrace{\int_0^{1-x-y}}_{Z} z\,dz\,dy\,dx$$





E parameterization

$0 \le x \le 1$

$0 \le y \le -x+1$

$0 \le z \le 1-x-y$

$$= \int_0^1 \int_0^{-x+1} \left.\frac{z^2}{2}\right|_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{-x+1} (1-x-y)^2\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{-x+1} (x^2+y^2+2xy-2x-2y+1)\,dy\,dx$$

$$(1-x-y)(1-x-y)$$
$$= 1-x-y-x+x^2+xy$$
$$-y+yx+y^2$$
$$= x^2+y^2+2xy-2x$$
$$-2y+1$$

$$= \frac{1}{2}\int_0^1 \left[x^2 y + \frac{y^3}{3} + \underbrace{2x\frac{y^2}{2}}_{xy^2} - 2xy \underbrace{- 2\frac{y^2}{2}}_{-y^2} + y \right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2(1-x) + \frac{1}{3}(1-x)^3 + x(1-x)^2 - 2x(1-x) - (1-x)^2 + (1-x)\; dx$$

$x \mid -\frac{2}{3} + 1 - 2 + 1$

$$= \frac{1}{2}\int_0^1 (1-x)\Big[x^2 + \underbrace{\frac{1}{3}(1-x)^2}_{\frac{1}{3}-\frac{2}{3}x+\frac{1}{3}x^2} + \underbrace{x(1-x)}_{x-x^2} - 2x \underbrace{-(1-x)}_{-1+x} + 1\Big)\, dx$$

$$= \frac{1}{2}\int_0^1 (1-x)\left[\frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3}\right] dx = \frac{1}{2}\int_0^1 \left(\frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3} - \frac{1}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{3}x\right) dx$$

$$\frac{1}{2}\int_0^1 -\frac{1}{3}x^3 + x^2 - x + \frac{1}{3}\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{3}\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{3}x\right]_0^1$$

$$= \frac{1}{2}\left[-\frac{1}{12} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\frac{-1+4-6+4}{12}\right] = \boxed{\frac{1}{24}}$$

Method 2

$$\frac{1}{2}\int_0^1\int_0^{-x+1} (1-x-y)^2\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 -\frac{(1-x-y)^3}{3}\Bigg|_{y=0}^{-x+1} dx$$

$$= \frac{1}{2}\int_0^1 \left[\underbrace{-\frac{1}{3}(1-x+x-1)^3}_{0} + \frac{1}{3}(1-x)^3\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{3}(1-x)^3\,dx = \frac{1}{6}\int_0^1 (1-x)^3\,dx$$

$$= \frac{1}{6}\int_1^0 -u^3\,du = -\frac{1}{6}\frac{u^4}{4}\Bigg|_1^0 = -\frac{1}{6}\left[0 - \frac{1}{4}\right]$$

$u = 1-x$

$du = -dx$

$$= \frac{1}{24}$$