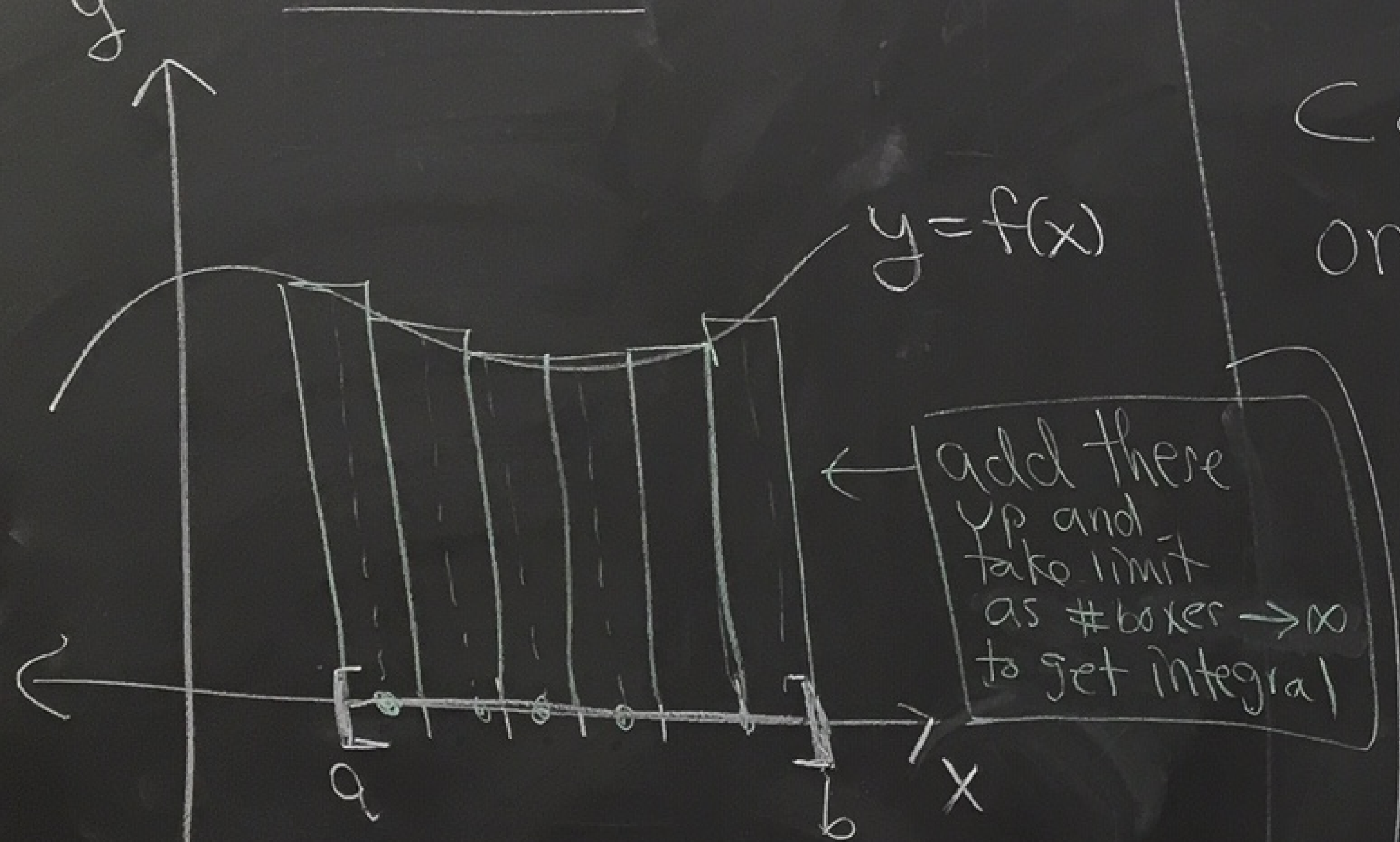
Thursday
10/3

13.1 - Double integrals over rectangular regions

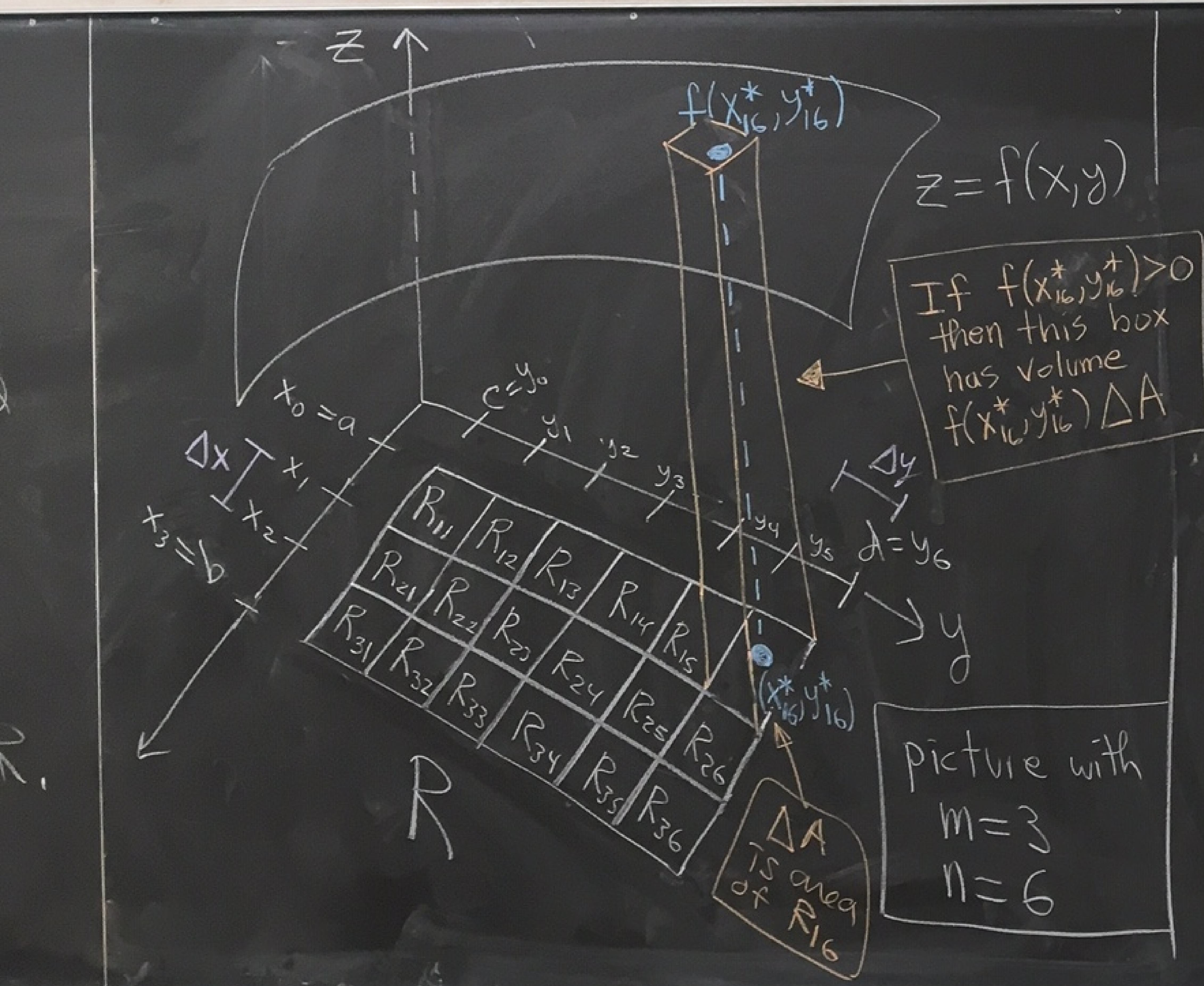
Calc I



Consider a function $f(x,y)$ defined on a closed rectangle

$$R = \left\{ (x,y) \;\middle|\; \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix} \right\}.$$

Let's define the integral of f over R.



① Divide R up into subrectangles by:

Divide $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$.

Divide $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$.

② Use these points to subdivide R into rectangles

$$R_{ij} = \left\{ (x,y) \;\middle|\; \begin{matrix} x_{i-1} \le x \le x_i \\ y_{j-1} \le y \le y_j \end{matrix} \right\}$$

each R_{ij} has area $\Delta A = \Delta x \, \Delta y$

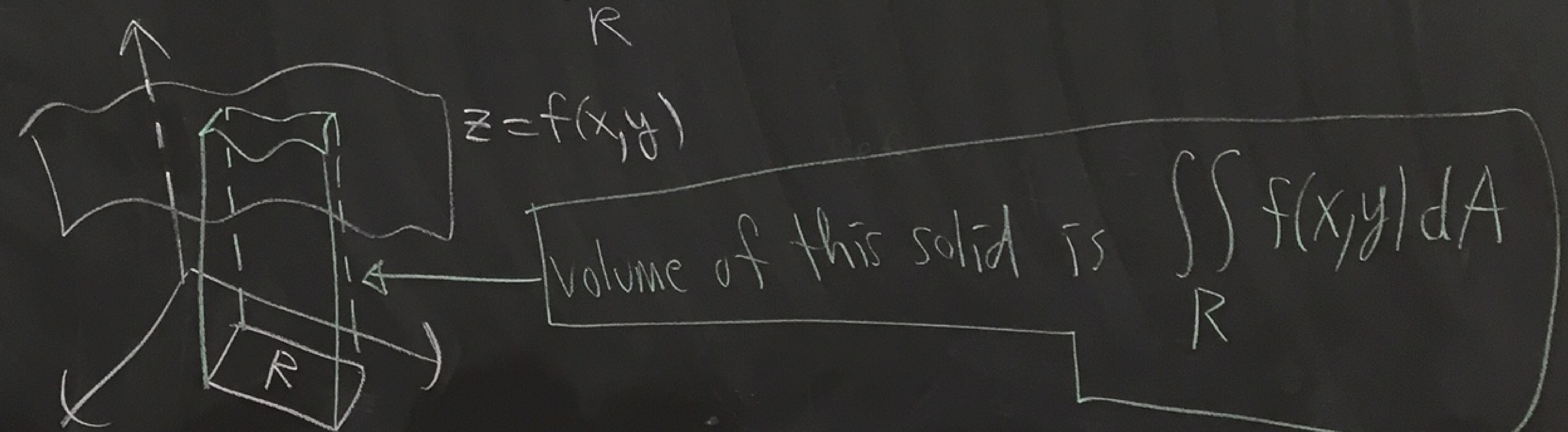
③ In each R_{ij} pick a point (x^*_{ij}, y^*_{ij}).

④ The double integral of f over the rectangle R is defined to be the following limit (if it exists).

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{f(x^*_{ij}, y^*_{ij})\,\Delta A}_{\text{if this is } >0, \text{ this is the volume of the box over } R_{ij}}$$

Facts:

- If f is continuous over R, then $\iint_R f(x,y)\,dA$ will exist.
- If $f(x,y) \geq 0$ over R, then the volume of the solid below the surface $z=f(x,y)$ and above R is $\iint_R f(x,y)\,dA$.



How do we calculate $\iint_R f(x,y)\,dA$?

We need iterated integrals!

We use the notation $\int_c^d f(x,y)\,dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from $y=c$ to $y=d$.

Similarly, $\int_a^b f(x,y)\,dx$ means to hold y fixed and integrate $f(x,y)$ with respect to x from $x=a$ to $x=b$.

Define

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx \quad \text{and} \quad \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

These are called <u>iterated integrals</u>.

Ex:

$$\int_0^1 \int_{-1}^2 (x^2y+x)\,dy\,dx$$

y bounds

x bounds

Integral formulas

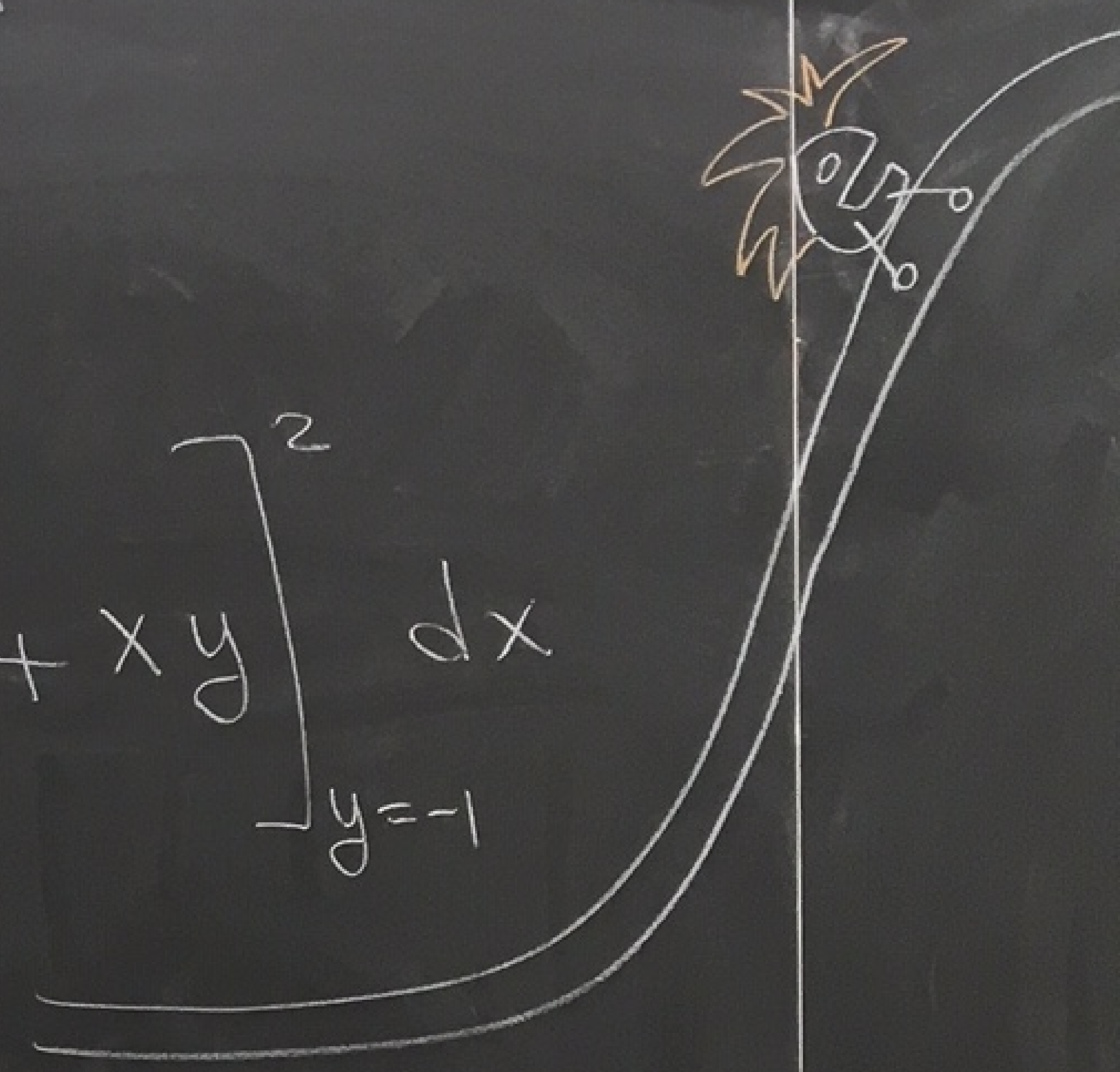
$\boxed{n \neq -1}$

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + C$$

$$\int \frac{1}{t}\,dt = \ln|t| + C$$

$$= \int_0^1 \left[\int_{-1}^2 (x^2y+x)\,dy\right] dx = \int_0^1 \left[x^2\frac{y^2}{2} + xy\right]_{y=-1}^{2} dx$$

$$= \int_0^1 \left[\underbrace{\left(\frac{x^2}{2}\cdot 2^2 + x(2)\right)}_{(2x^2+2x)} - \underbrace{\left(\frac{x^2}{2}(-1)^2 + x(-1)\right)}_{\left(\frac{x^2}{2}-x\right)}\right] dx$$



$$\int_0^1 \left(\frac{3}{2}x^2+3x\right)dx$$

$$=\left[\frac{3}{2}\frac{x^3}{3}+3\frac{x^2}{2}\right]_{x=0}^{1}$$

$$=\left[\left(\frac{3}{2}\frac{(1)^3}{3}+\frac{3}{2}(1)^2\right)-\left(\frac{3}{2}\frac{(0)^3}{3}+\frac{3(0)^2}{2}\right)\right]=\frac{1}{2}+\frac{3}{2}=\boxed{2}$$

<u>Fubini's Theorem</u> If $f(x,y)$ is continuous on the rectangle $R = \left\{ (x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$

then $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$