$(ab)^{-1} = b^{-1}a^{-1}$

$(ncn^{-1})^{-1} = (n^{-1})^{-1}c^{-1}n^{-1}$
$= nc^{-1}n^{-1}$

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4.4 continued...

Theorem Let G be a group and $H \leq G$. Then, $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.

$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$

$C_G(H) = \{g \in G \mid \underbrace{gh = hg}_{ghg^{-1} = h} \text{ for all } h \in H\}$

$C_G(H) \leq N_G(H)$

$\boxed{C_G(H) \trianglelefteq N_G(H)}$

Proof: Let $c \in C_G(H)$ and $n \in N_G(H)$. We need to show $ncn^{-1} \in C_G(H)$.
Let $h \in H$.
Since $h \in H$ and $n \in N_G(H)$, we know $n^{-1}hn \in H$. $\boxed{nHn^{-1} = H}$
Since $c \in C_G(H)$ we have $c(n^{-1}hn)c^{-1} = n^{-1}hn$.
So, $(ncn^{-1})h(ncn^{-1})^{-1} = nc(n^{-1}hn)c^{-1}n^{-1}$
$= nn^{-1}hnn^{-1} = h$
So, $ncn^{-1} \in C_G(H)$. ∎

Claim:

Then,

An

$= G$.

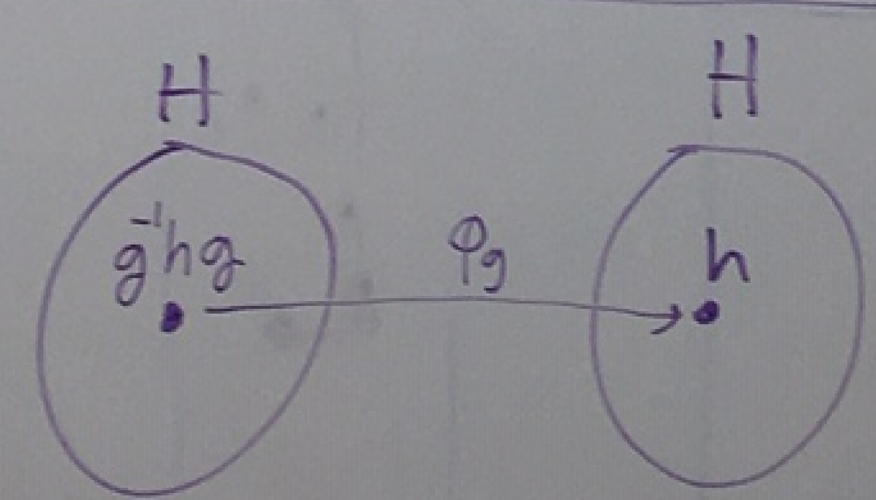
Proof: Given $g \in N_G(H)$ define $\varphi_g : H \to H$ by $\varphi_g(h) = ghg^{-1}$.

This makes sense because $g \in N_G(H)$ so $gHg^{-1} = H$

Claim: φ_g is an isomorphism from H to H if $g \in N_G(H)$

1-1: Suppose $\varphi_g(x_1) = \varphi_g(x_2)$.
Then $g x_1 g^{-1} = g x_2 g^{-1}$.
So, $g^{-1}(g x_1 g^{-1}) g = g^{-1}(g x_2 g^{-1}) g$.
Thus, $x_1 = x_2$.

onto:
Since $g \in N_G(H)$ and $N_G(H)$ is a group, we know $g^{-1} \in N_G(H)$.
Let $h \in H$.
Then, $g^{-1} h g = \underbrace{g^{-1} h (g^{-1})^{-1}}_{\text{since } g^{-1} \in N_G(H)} \in H$.
And $\varphi_g(g^{-1} h g) = g(g^{-1} h g) g^{-1} = h$.



homomorphism:
Let $h_1, h_2 \in H$.
Then,
$$\varphi_g(h_1 h_2) = g h_1 h_2 g^{-1} = g h_1 g^{-1} g h_2 g^{-1}$$
$$= \varphi_g(h_1)\varphi_g(h_2).$$

Claim

So if $g \in N_G(H)$ then $\varphi_g \in \mathrm{Aut}(H)$.

Define $\Psi : N_G(H) \longrightarrow \mathrm{Aut}(H)$

by $\Psi(g) = \varphi_g$.

<u>Ψ is a homomorphism</u>

Let $g_1, g_2 \in N_G(H)$.

We need to show $\varphi_{g_1g_2} = \Psi(g_1g_2) = \Psi(g_1) \circ \Psi(g_2) = \varphi_{g_1} \circ \varphi_{g_2}$.

Let $h \in H$.

Then $(\varphi_{g_1} \circ \varphi_{g_2})(h) = \varphi_{g_1}(\varphi_{g_2}(h)) = \varphi_{g_1}(g_2 h g_2^{-1}) = g_1 g_2 h g_2^{-1} g_1^{-1} = (g_1 g_2) h (g_1 g_2)^{-1}$

$= \varphi_{g_1 g_2}(h)$.

So, $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2}$.

$\operatorname{Ker}(\psi) = C_G(H)$

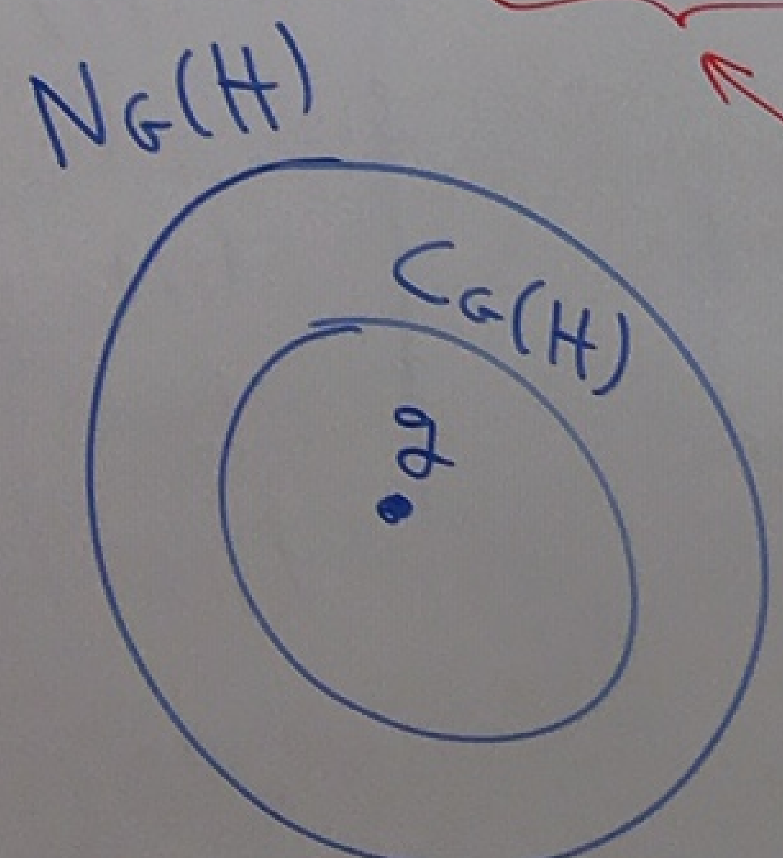
$$\operatorname{Ker}(\psi) = \{ g \in N_G(H) \mid \varphi_g = i \}$$

$\boxed{\begin{array}{l} i : H \to H \\ i(h) = h \ \ \forall h \in H \end{array}}$

$$= \{ g \in N_G(H) \mid \varphi_g(h) = h \text{ for all } h \in H \}$$

$$= \{ \underbrace{g \in N_G(H)}_{g \in N_G(H)} \mid \underbrace{ghg^{-1} = h \text{ for all } h \in H}_{g \in C_G(H)} \} = C_G(H)$$

$\boxed{C_G(H) = \{ g \in G \mid ghg^{-1} = h \text{ for all } h \in H \}}$



By the First Isomorphism Theorem,

$$N_G(H)/C_G(H) = N_G(H)/\mathrm{Ker}(\psi) \cong \mathrm{im}(\psi) \leq \mathrm{Aut}(H)$$

$\boxed{\psi : N_G(H) \to \mathrm{Aut}(H)}$

Corollary: If G is a group and $H \trianglelefteq G$ then $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

proof: If $H \trianglelefteq G$, then $gHg^{-1} = H$ for all $g \in G$.

So, $G = \{g \in G \mid gHg^{-1} = H\} = N_G(H)$.

4.5 - Sylow's Theorem

Def: Let G be a group. Let p be a prime.

1. A group of size p^{α} where $\alpha \geq 0$ is called a p-group
2. If G is a p-group and $H \leq G$, then H is called a p-subgroup of G
3. If $|G| = p^{\alpha} m$ where $p \nmid m$, then a subgroup of G of size p^{α} is called Sylow p-subgroup of G.
4. Suppose $|G| = p^{\alpha} m$ where $p \nmid m$. Define $Syl_p(G)$ to be the set of all sylow p-subgroups of G.

The size of $Syl_p(G)$ is denoted by $n_p(G)$ or just n_p.

Ex: $\mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$

$|\mathbb{Z}_{12}| = 2^2 \cdot 3$

3-subgroups	size
$\{\bar{0}\}$	3^0
$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}, \bar{8}\}$	3^1

3-sylow subgroup

$Syl_3(\mathbb{Z}_{12}) = \{\langle \bar{4} \rangle\}$

$n_3(\mathbb{Z}_{12}) = 1$

2-subgroups	size
$\{\bar{0}\}$	2^0
$\{\bar{0}, \bar{6}\}$	2^1
$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$	2^2

2-sylow subgroup

$Syl_2(\mathbb{Z}_{12}) = \{\langle \bar{3} \rangle\}$

$n_2(\mathbb{Z}_{12}) = 1$