Thursday
10/3

HW 2.2

6 (a) Given $H \leq G$.

Prove $H \leq N_G(H)$.

Recall $N_G(H) = \{ g \in G \mid gHg^{-1} = H \}$

proof: Let $h \in H$. We will show $hHh^{-1} = H$ which will give us that $h \in N_G(H)$.

$\boxed{\supseteq}$: Let $x \in H$.

Since $h \in H$ and H is a subgroup we know $h^{-1} \in H$.

Since $h, h^{-1}, x \in H$ and H is a subgroup we know $h^{-1}xh \in H$.

Then, $x = h(h^{-1}xh)h^{-1} \in hHh^{-1}$.

So, $H \subseteq hHh^{-1}$.

since $h^{-1}xh \in H$

$\boxed{\subseteq}$: Let's show $hHh^{-1} \subseteq H$.

Let $y \in hHh^{-1}$.

Then $y = h\hat{h}h^{-1}$ where $\hat{h} \in H$.

Since $h, \hat{h}, h^{-1} \in H$ and H is a subgroup we know $y = h\hat{h}h^{-1} \in H$.

So, $hHh^{-1} \subseteq H$.

By $\boxed{\supseteq}$ and $\boxed{\subseteq}$ we have that $H = hHh^{-1}$.

Thus, $H \leq N_G(H)$.

3.5 continued...

Thm: For any $\sigma \in S_n$ there may be many ways of writing σ as a product of transpositions; however if you express σ in two different ways as the product of transpositions then either both expressions contain an even number of transpositions or both expressions contain an odd number of transpositions.

Ex: In S_4,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \underbrace{(2,4)}_{\text{1 transpositions}} = \underbrace{(4,3)(2,1)(3,2)(1,4)(3,1)}_{\text{5 transpositions}}$$

Def: Define the function

$$\varepsilon : S_n \longrightarrow \{1,-1\}$$

by

$$\varepsilon(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ can be expressed as a product of an } \underline{\text{even}} \text{ \# of transpositions} \\ -1, & \text{if } \sigma \text{ can be expressed as a product of an } \underline{\text{odd}} \text{ \# of transpositions} \end{cases}$$

Think of $\{1,-1\}$ as a subgroup of $\mathbb{R}-\{0\}$ under multiplication

ie, $\{1,-1\}$ is a group under multiplication

We say

if $\varepsilon(\sigma$

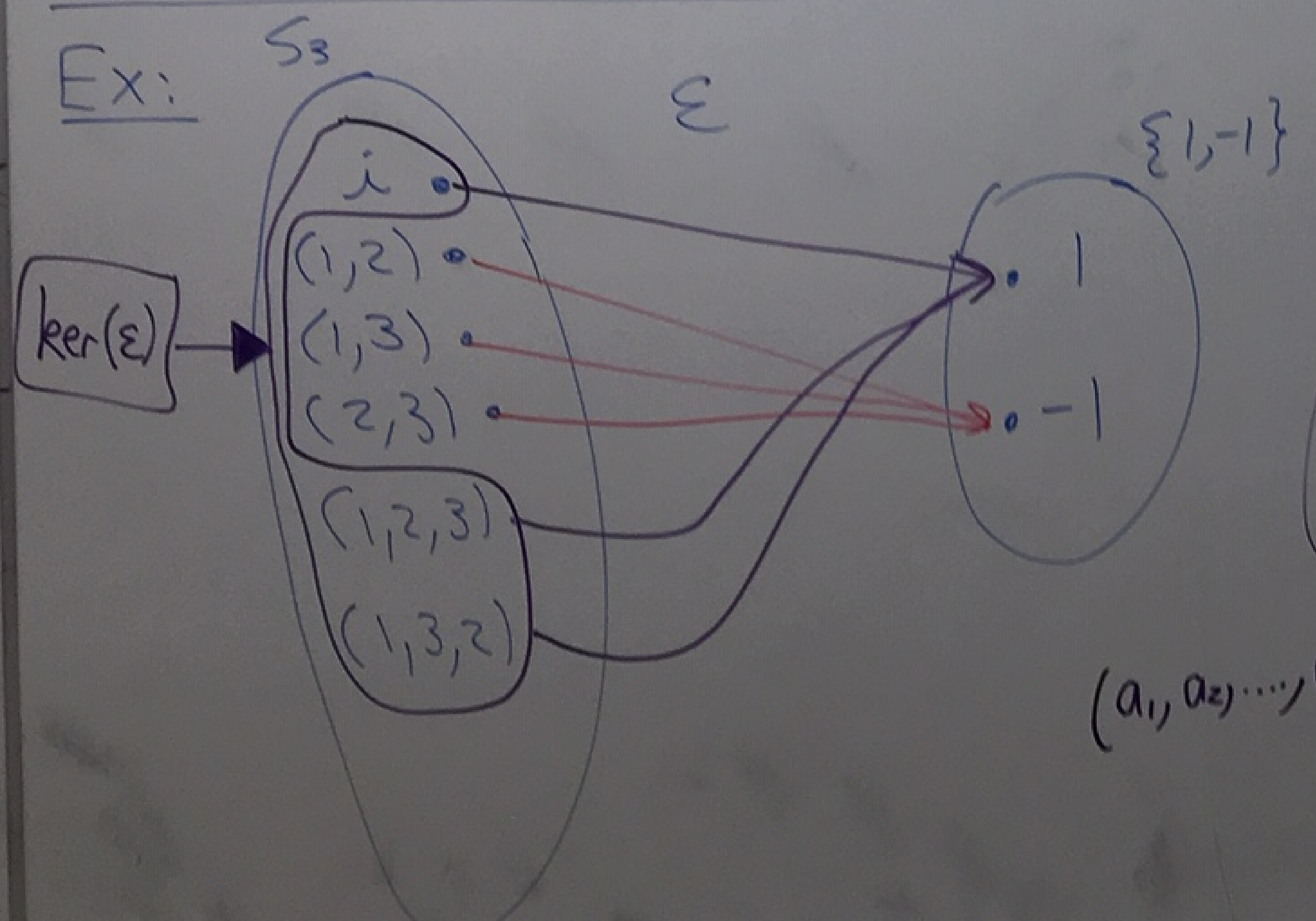
if $\varepsilon(\sigma$

Ex:

$\boxed{\ker(\varepsilon)} \rightarrow$

We say that σ is an <u>even permutation</u> if $\varepsilon(\sigma)=1$ and σ is an <u>odd permutation</u> if $\varepsilon(\sigma)=-1$.

<u>Ex:</u>



$$\boxed{\begin{aligned} i &= (1,2)(1,2) \\ (1,2,3) &= (1,3)(1,2) \\ (1,3,2) &= (1,2)(1,3) \end{aligned}}$$

$$(a_1, a_2, \ldots, a_m) = (a_1, a_m)(a_1, a_{m-1}) \cdots (a_1, a_2)$$

<u>Theorem:</u> $\varepsilon : S_n \to \{1,-1\}$ is a homomorphism.

<u>proof:</u> Let $\sigma, \tau \in S_n$.

We need to show that

$$\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau).$$

<u>case1:</u> Suppose σ is even and τ is even.

So, $\varepsilon(\sigma) = \varepsilon(\tau) = 1$.

Since σ is even, $\sigma = t_1 t_2 \cdots t_{2k}$ where t_i is a transposition.

Since τ is even, $\tau = s_1 s_2 \cdots s_{2\ell}$ where s_i is a transposition.

Then $\sigma\tau = t_1 t_2 \cdots t_{2k} s_1 s_2 \cdots s_{2\ell}$ is the product of $2k+2\ell$ transpositions.

And $2k+2\ell$ is even.

So, $\varepsilon(\sigma\tau) = 1$.

So, $\varepsilon(\sigma\tau) = 1 = \varepsilon(\sigma)\varepsilon(\tau)$

The other cases are similar.
[σ odd, τ odd / σ odd, τ even / σ even, τ odd]

So, ε is a homomorphism. ▨

Def: The alternating group A_n is defined to be the kernel of $\varepsilon : S_n \to \{1,-1\}$.
That is, A_n is the subgroup of S_n consisting of all even permutations.

Facts:

① $A_n \trianglelefteq S_n$

pf: $A_n = \ker(\varepsilon) \trianglelefteq S_n$. ▨

② $S_n/A_n \cong \{1,-1\} \cong \mathbb{Z}_2$

pf: $\varepsilon : S_n \to \{1,-1\}$ is onto so by the first isomorphism thm
$S_n/A_n = S_n/\ker(\varepsilon) \cong \operatorname{im}(\varepsilon) = \{1,-1\}$ ▨

③ $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

pf: By Lagrange and ②, $|S_n| = \binom{\#\text{ left}}{\text{cosets}} \cdot |A_n|$ $= |S_n/A_n$ ...

So, $|A_n$ ...

Ex:

$A_3 = \{$ ...

$= |S_n/A_n| \cdot |A_n| = 2|A_n|.$

So, $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$ $\blacksquare$

Ex:

$A_3 = \{ i, (1,2,3), (1,3,2) \}$

Cayley's Thm

Def: Let S be a set. The automorphism group on S is $Aut(S) = \{ f : S \to S \mid f \text{ is a bijection} \}$

Ex: $S = \{1, 2, \ldots, n\}$

$Aut(S) = S_n$

Fact: $Aut(S)$ is a group.

- Given $\tau, \sigma \in Aut(S)$, the group operation is $\tau\sigma = \tau \circ \sigma$
- The identity function $i : S \to S$ given by $i(s) = s$ $\forall s \in S$ is the identity in the group.
- Given $\sigma \in Aut(S)$, the inverse function σ^{-1} is the inverse of σ in the group.

If $|G| = n$, then $\mathrm{Aut}(G) \cong S_n$

Cayley's thm: Let G be a group. Then G is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

pf: Define $\Psi : G \to \mathrm{Aut}(G)$ by $\Psi(g) = \tau_g$ where $\tau_g : G \to G$ is given by $\tau_g(a) = ga$.

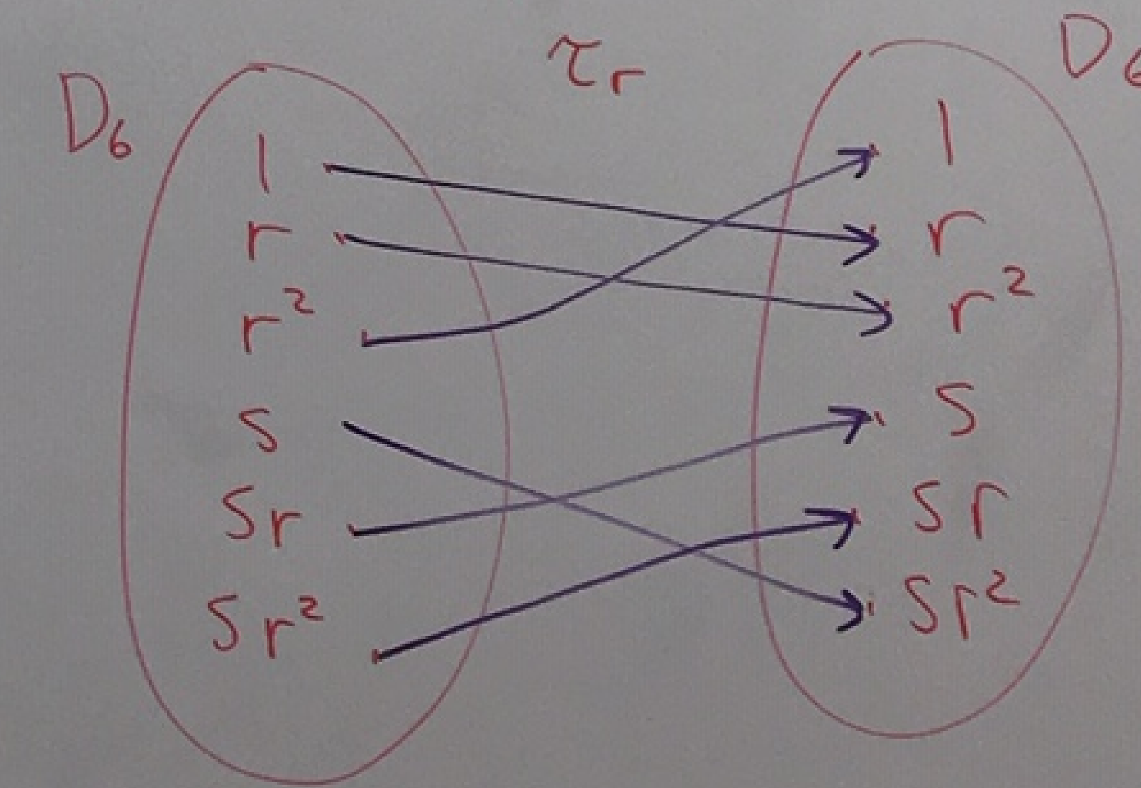
Need to prove:

1. $\tau_g \in \mathrm{Aut}(G)$ for all $g \in G$.
2. Ψ is a homomorphism, that is $\Psi(gh) = \Psi(g) \circ \Psi(h)$ for all $g, h \in G$.
3. Ψ is one-to-one.

Ex: $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

$\Psi : D_6 \to \mathrm{Aut}(D_6)$

$\Psi(r) = \tau_r$ where $\tau_r(a) = ra$.



$rs = sr^2$

$,s,sr,sr^2\}$

$\tau_r(a) = ra.$

D_6

1
r
r^2
s
sr
sr^2

Using ①, ②, ③ we have that

$$G \cong \mathrm{im}(\psi) \leq \mathrm{Aut}(G). \quad \blacksquare$$