11/21 Thursday

$\boxed{5.4}$

Motivation:

Given G we want a normal subgroup H, where G/H is abelian.

So we want
$$(xH)(yH) = (yH)(xH) \quad \forall x,y \in G$$

ie
$$xyH = yxH \quad \forall x,y \in G$$

ie
$$x^{-1}y^{-1}xyH = H \quad \forall x,y \in G$$

ie $x^{-1}y^{-1}xy \in H \quad \forall x,y \in G$

Def: Let G be a group.

Define
$$G' = \langle \{ x^{-1}y^{-1}xy \mid x,y \in G \} \rangle$$

So G' is the subgroup of G that is generated by all the elements of the form $x^{-1}y^{-1}xy$. G' is called the commutator subgroup of G.

Ex: $D_6 = \{1, r, r^2, s, sr, sr^2\}$

x	y	$x^{-1}y^{-1}xy$
1	a	1
a	1	1
a	a	1
r	s	$r^{-1}s^{-1}rs = r^{-2} = r$
s	r	$s^{-1}r^{-1}sr = r^2$
r	r^2	1
r^2	r	1
r	sr	$r^{-1}(sr)^{-1}rsr = r^2srrsr = r$
r	sr^2	$r^{-1}(sr^2)^{-1}rsr^2 = r^2sr^2rsr^2 = r$
r^2	s	r^2
r^2	sr	r^2
r^2	sr^2	r^2

x	y	$x^{-1}y^{-1}xy$
s	sr	r^2
s	sr^2	r
s	r	r^2
s	r^2	r
sr	r	r^2
sr	r^2	r
sr	s	r
sr	sr^2	r^2
sr^2	r	r^2
sr^2	r^2	r
sr^2	s	r^2
sr^2	sr	r

$$(D_6)' = \langle \{1, r, r^2\} \rangle = \{1, r, r^2\}$$

$$D_6/(D_6') = \{D_6', sD_6'\} \cong \mathbb{Z}_2$$

So D_6/D_6' is abelian.

$D_6' = \{1, r, r^2\}$

$sD_6' = \{s, sr, sr^2\}$

Theorem: G is abelian iff $G' = \{1\}$

proof:

G is abelian iff $xy = yx$ for all $x, y \in G$

iff $x^{-1}y^{-1}xy = 1$ for all $x, y \in G$

iff $\{x^{-1}y^{-1}xy \mid x, y \in G\} = \{1\}$

iff $G' = \{1\}$.

Theorem: Let G be a group.

1. $G' \trianglelefteq G$.
2. G/G' is abelian.
3. If $H \trianglelefteq G$ and G/H is abelian then $G' \leq H$.

Conversely, if $G' \leq H \leq G$ then $H \trianglelefteq G$ and G/H is abelian.

G/G' is the "largest" abelian quotient of G.

- If G/H is abelian, then $G' \leq H$

So, $\left|G/H\right| = \frac{|G|}{|H|} < \frac{|G|}{|G'|} = \left|G/G'\right|$

- If G/H is abelian then $G' \leq H$.

Proof:

① For later.

② Let $x, y \in G$. Then

$$(xG')(yG') = (xy)G' = [(xy)G'][(y^{-1}x^{-1}yx)G']$$

$y^{-1}x^{-1}yx \in G'$

$(y^{-1}x^{-1}yx)G' = G'$

$$= (xyy^{-1}x^{-1}yx)G' = (yx)G' = (yG')(xG').$$

So, G/G' is abelian.

③ ($\Longrightarrow$) Suppose $H \trianglelefteq G$ and G/H is abelian. We will show $G' \leq H$.

Since G/H is abelian $(xH)(yH) = (yH)(xH)$ for all $x, y \in G$.

So, $(x^{-1}H)(y^{-1}H)(xH)(yH) = H$ for all $x, y \in G$.

Merging these we get $(x^{-1}y^{-1}xy)H = H$ for all $x, y \in G$.

So, $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$.

So all the generators of G' are contained in H.

Thus, $G' = \langle \{x^{-1}y^{-1}xy \mid x, y \in G\} \rangle \leq H$.

($\Leftarrow$) Suppose $G' \leq H \leq G$.

We want to show $H \trianglelefteq G$ and G/H is abelian.

<u>$H \trianglelefteq G$</u>: Let $g \in G$ and $h \in H$.

Then $ghg^{-1} = ghg^{-1}h^{-1}h = g'h$ where $g' = ghg^{-1}h^{-1} \in G'$.

Since $g' \in G' \leq H$ and $h \in H$ we get $ghg^{-1} = g'h \in H$.

So, $H \trianglelefteq G$.

<u>G/H is abelian:</u> Let $x, y \in G$. Then

$$(xH)(yH) = (xy)H = \left[((xy)H)\big(\underbrace{(y^{-1}x^{-1}yx)H}_{H}\big)\right] = (xyy^{-1}x^{-1}yx)H = yxH = (yH)(xH).$$

$y^{-1}x^{-1}yx \in G' \leq H$
So, $y^{-1}x^{-1}yxH = H$

1. Let $g \in G$ and $g' \in G'$.

Then $gg'g^{-1} = \underbrace{gg'g^{-1}(g')^{-1}}_{\text{in } G'}\underbrace{g'}_{\text{in } G'} \in G'$.

So, $G' \trianglelefteq G$.

$\blacksquare$

Ex: Classify the groups of size 45 up to isomorphism.

Let G be a group of size $45 = 3^2 \cdot 5$.

Claim: G is abelian.

Let P_3 be a Sylow 3-subgroup of G, that is $|P_3| = 3^2 = 9$.

Let P_5 be a Sylow 5-subgroup of G, that is $|P_5| = 5$

We know $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 5$. So, $n_3 = 1$.

We know $n_5 \equiv 1 \pmod{5}$ and $n_5 \mid 9$. So, $n_5 = 1$.

$\times \mathbb{Z}_3$

$\mathbb{Z}_3$

Since $n_3 = 1$ we know $P_3 \trianglelefteq G$.
Since $n_5 = 1$ we know $P_5 \trianglelefteq G$.

We know $P_3 \cap P_5 \leq P_3$ and $P_3 \cap P_5 \leq P_5$.
So, $|P_3 \cap P_5|$ divides $|P_3| = 9$ and $|P_5| = 5$, by Lagrange.

Thus, $|P_3 \cap P_5| = 1$.
So, $P_3 \cap P_5 = \{1\}$.

Since $\left|G/P_3\right| = \dfrac{|G|}{|P_3|} = \dfrac{3^2 \cdot 5}{3^2} = 5$ and $P_3 \trianglelefteq G$ we have that G/P_3 is a cyclic group.
prime

So, G/P_3 is abelian.
Thus, $G' \leq P_3$. [By 3 of previous thm.]

nge.

is a cyclic group.

Since $\left|G/P_5\right| = \frac{45}{5} = 3^2$ and $P_5 \trianglelefteq G$

$(\text{prime})^2$

we have G/P_5 is abelian.

So, $G' \leq P_5$ [③ of previous thm.]

Thus, $G' \leq P_3 \cap P_5 = \{1\}$.

So, $G' = \{1\}$.

Thus, G is abelian.

Since G is abelian of size $45 = 3^2 \cdot 5$

$\boxed{G \cong \mathbb{Z}_{3^2 \cdot 5} = \mathbb{Z}_{45}}$ or $\boxed{\begin{aligned} G &\cong \mathbb{Z}_{3\cdot 5} \times \mathbb{Z}_3 \\ &= \mathbb{Z}_{15} \times \mathbb{Z}_3 \end{aligned}}$

Ex:

up t

Let

Claim:

Let P_3

Let P_5

We know

We know