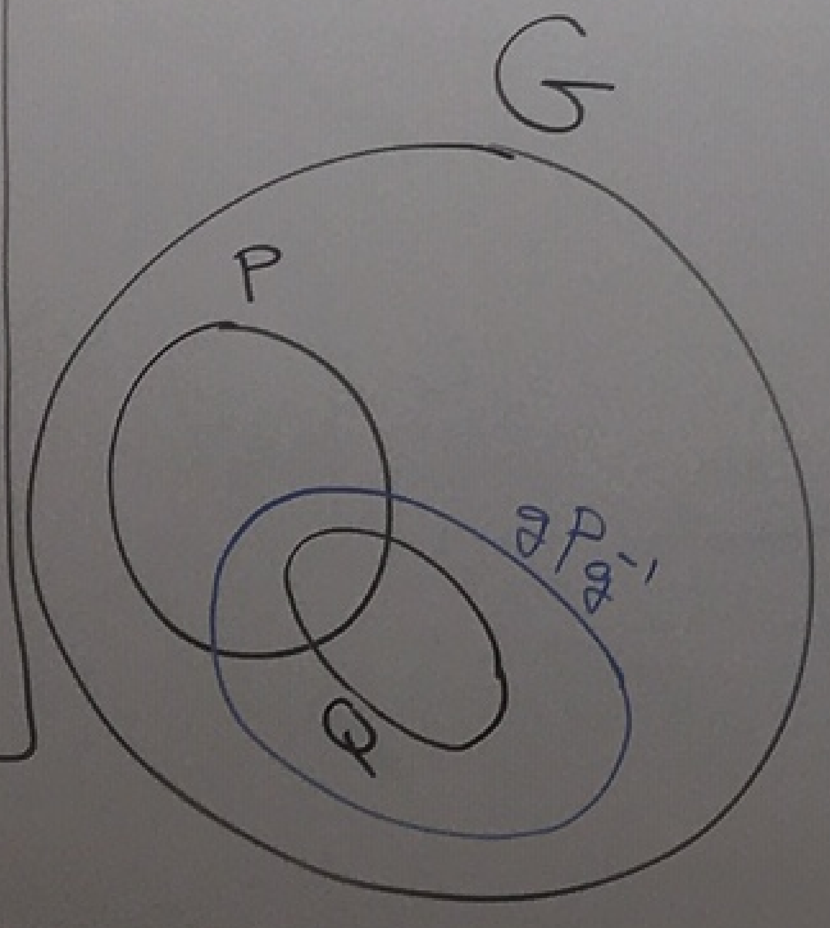
11/5
Tuesday

Fact: Let G be a finite group and $H \leq G$.
Given $g \in G$, $gHg^{-1} \leq G$ and $|gHg^{-1}| = |H|$.



Sylow's Theorem Let G be a group with $|G| = p^{\alpha} m$ where p is prime and $p \nmid m$.

1. Sylow p-subgroups of G exist.
   That is, there is a subgroup of G of size p^{α}.
2. If P is a Sylow p-subgroup of G $[P \leq G$ and $|P| = p^{\alpha}]$ and Q is any p-subgroup of G, $[Q \leq G$ and $|Q| = p^{\beta}$ where $0 \leq \beta \leq \alpha]$ then there exists $g \in G$ where $Q \leq gPg^{-1}$.
   In particular, if $|Q| = p^{\alpha}$, then there exists $g \in G$ where $Q = gPg^{-1}$. So any two Sylow p-subgroups are conjugate to each other.

③

③ Recall that $n_p(G)$ is the number of Sylow p-subgroups of G.
Then $n_p(G) \equiv 1 \pmod{p}$
and $n_p(G)$ divides m.

p divides $n_p(G) - 1$

$n_p(G)$ is of the form $1 + pk$

Ex: $D_{12} = \{1, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$

$|D_{12}| = 12 = 2^2 \cdot 3 = p^\alpha \cdot m$

$P = \{1, r^3, s, sr^3\} \leq D_{12}$

P is a Sylow 2-subgroup of D_{12} since $|P| = 2^2$.

Part ③ gives us a prediction of what $n_2 = n_2(D_{12})$ is.

$n_2 \equiv 1 \pmod{2}$

$\to n_2 = 1, 3, 5, 7, 9, \dots$

n_2 divides 3.

$\boxed{n_2 = 1 \text{ or } n_2 = 3}$

All Sylow 2-subgroups are of the form gPg^{-1} where $g \in D_{12}$ by part ② of Sylow's Theorem.

$\boxed{P = \{1, r^3, s, sr^3\}}$

$rPr^{-1} = \{r \cdot 1 \cdot r^{-1}, r \cdot r^3 \cdot r^{-1}, r \cdot s \cdot r^{-1}, r \cdot sr^3 \cdot r^{-1}\}$

$\boxed{rPr^{-1} = \{1, r^3, sr^4, sr\}}$

$r^2P(r^2)^{-1} = \{r^2 \cdot 1 \cdot r^{-2}, r^2 \cdot r^3 \cdot r^{-2}, r^2 \cdot s \cdot r^{-2}, r^2 \cdot sr^3 \cdot r^{-2}\}$

$\boxed{r^2P(r^2)^{-1} = \{1, r^3, sr^2, sr^5\}}$

So, $n_2 = 3$. These are the 3 Sylow 2-subgroups.

$r^2 s r^3 r^{-2} = s r^{-2} r^3 r^{-2} = s r^{-1} = s r^5$

$r^6 = 1$

$sr^{-2} = sr^{6-2} = sr^4$

$rsr^3r^{-1} = sr^{-1}r^3r^{-1} = sr$

$r^{-1} = r^5$

$r^2 s r^{-2} = s r^{-2} r^{-2} = s r^{6-4} = sr^2$

Let's look at part 2 of Sylow's theorem more closely for D_{12}.

Let's pick a 2-subgroup of D_{12} that isn't size 4.

Let $Q = \{1, sr^5\}$. Then Q is a 2-subgroup of D_{12}.

Part 2 says that $Q \leq gPg^{-1}$ for some $g \in D_{12}$.

In this case, $Q \leq (r^2)P(r^2)^{-1}$.

Ex:

$m=4 \quad p^{\alpha}=3^{1}$

$|D_{12}| = 2^{2} \cdot 3$

$n_3 \equiv 1 \pmod{3}$

$n_3 = 1, 4, 7, 10, 13, \dots$

n_3 divides 4.

So, $n_3 = 1$ or $n_3 = 4$.

$P = \{1, r^2, r^4\} = \langle r^2 \rangle$ is a Sylow 3-subgroup.

$r^i P (r^i)^{-1} = \{r^i 1 r^{-i}, r^i r^2 r^{-i}, r^i r^4 r^{-i}\} = \{1, r^2, r^4\} = P$ for $i = 0,1,2,3,4,5$

$(sr^i) P \underbrace{(sr^i)^{-1}}_{r^{-i}s} = \{sr^i \cdot 1 \cdot r^{-i}s,\ sr^i r^2 r^{-i} s,\ sr^i r^4 \cdot r^{-i} s\}$

$= \{1, sr^2 s, sr^4 s\} = \{1, ssr^{-2}, ssr^{-4}\}$

$= \{1, r^{-2}, r^{-4}\} = \{1, r^4, r^2\} = P$

for $i = 0,1,2,3,4,5$.

So, $P = \{1, r^2, r^4\}$ is the only Sylow 3-subgroup of D_{12}.
So, $n_3 = 1$. Also, this means $P \trianglelefteq D_{12}$ since $gPg^{-1} = P$ for $g \in D_{12}$.

Proof of 1 : We induct on $|G|$.

(base case) If $|G|=2$. Then, $G=\{1,x\}$ where $x\neq 1$.
The only prime $p \mid |G|$ is $p=2$.
In this case $P=G$ is a Sylow 2-subgroup of G.
The theorem is true if $|G|=2$.

Now suppose $|G|>2$.
(induction hypothesis) Assume the theorem holds for all groups of size smaller than $|G|$.

Let $|G|=p^{\alpha}m$ where p is a prime, $\alpha\geq 1$, $p\nmid m$.

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Case 1: Suppose $p \mid |Z(G)|$.

By Cauchy, there exists an element $z \in Z(G)$ with $|z| = p$.

Let $N = \langle z \rangle$.

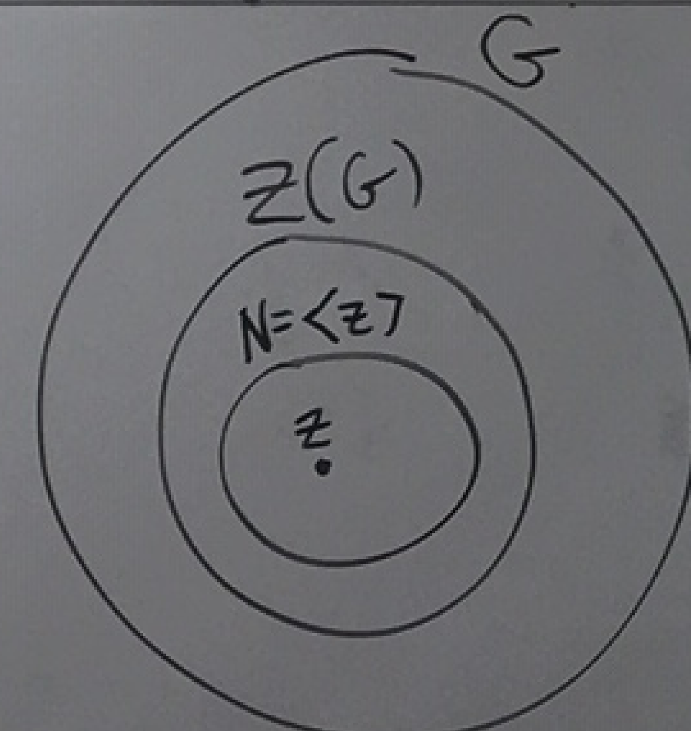
Then, $|N| = p$ and $N \leq Z(G)$.

Since $N \leq Z(G)$ we have

$$gN = \{gn \mid n \in N\} \underset{\boxed{N \leq Z(G)}}{=} \{ng \mid n \in N\} = Ng$$

for all $g \in G$.

So, $N \trianglelefteq G$.



Then G/N is a group.

And $|G/N| = \dfrac{|G|}{|N|} = \dfrac{p^{\alpha} m}{p} = p^{\alpha-1} m < |G|$.

So by the induction hypothesis, there exists a Sylow p-subgroup of G/N.

Let $\bar{P} \leq G/N$ where $|\bar{P}| = p^{\alpha-1}$. $\boxed{|G/N| = p^{\alpha-1} \cdot m, \quad p \nmid m}$

Consider the homomorphism $\pi: G \to G/N$ given by $\pi(g) = gN$.

Let $P = \pi^{-1}(\bar{P}) = \{g \in G \mid \pi(g) \in \bar{P}\}$

$= \{g \in G \mid gN \in \bar{P}\}$.

By the correspondence theorem, we have that $P \leq G$ and $N \leq P$.

Suppose that

$$\overline{P} = \{N, g_2N, g_3N, \ldots, g_{p^{\alpha-1}}N\}$$

where $g_i \in G$.

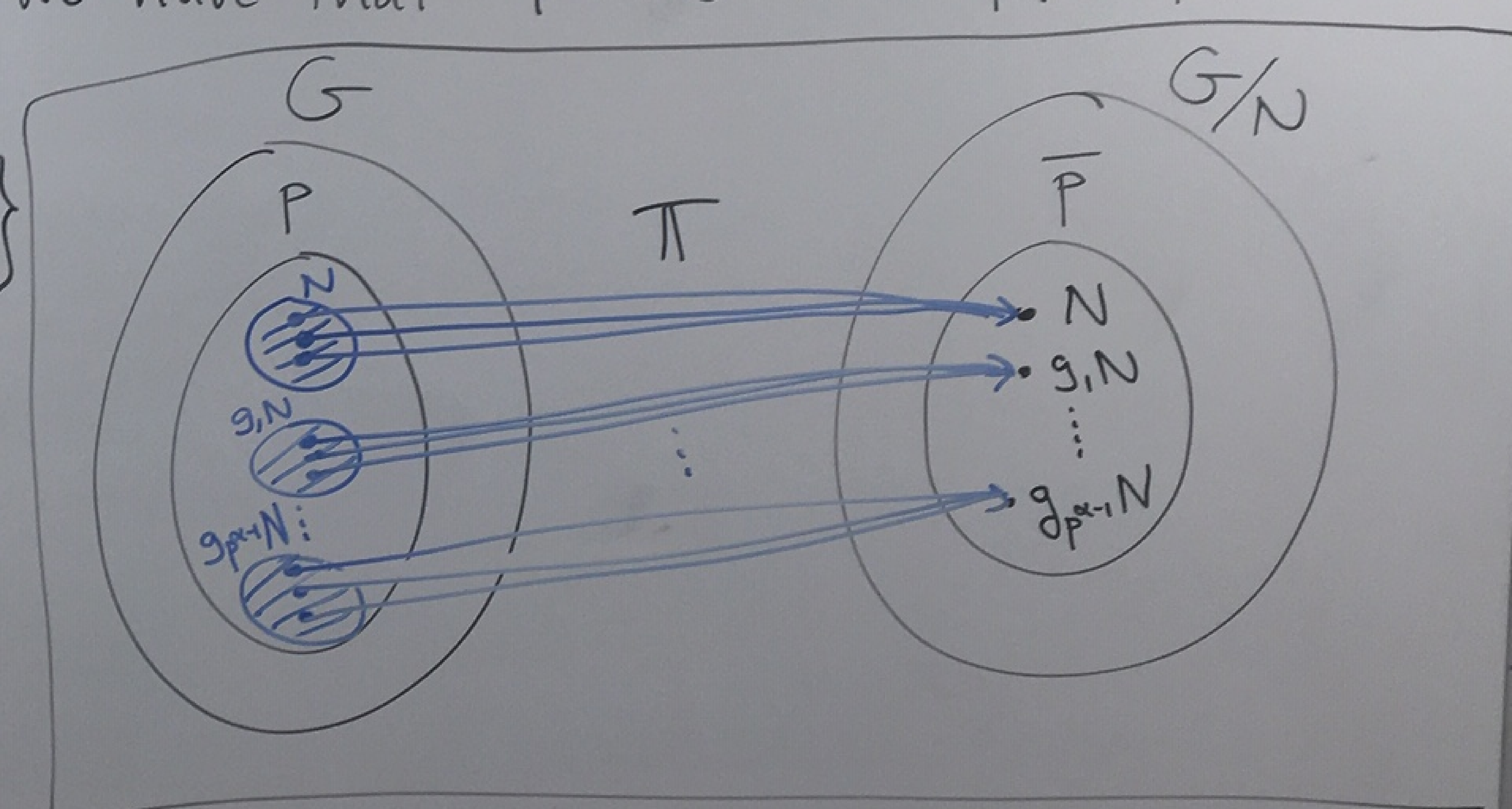
Then

$$P = N \cup g_2N \cup \cdots \cup g_{p^{\alpha-1}}N$$

Since these cosets are disjoint and all of size N we have that

$$|P| = |N| + |g_2N| + \cdots + |g_{p^{\alpha-1}}N|$$

$$= \underbrace{p + p + \cdots + p}_{p^{\alpha-1} \text{ of these}} = p(p^{\alpha-1}) = p^{\alpha}.$$



So, P is a Sylow p-subgroup of G.

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