12/3
Tuesday

14.2 continued...

Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by $\vec{r}(t)$ with $a \leq t \leq b$.

The line integral of $\vec{F}$ along C is

$$\int_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}_{\text{this is written } \int_C \vec{F}\cdot d\vec{r}}$$

$\vec{T}$ is the unit tangent vector on C

If $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$ and $\vec{r}(t) = \langle x(t), y(t)\rangle$, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \underbrace{\langle f(x(t),y(t)), g(x(t),y(t))\rangle}_{\vec{F}(\vec{r}(t))}\cdot\underbrace{\langle x'(t), y'(t)\rangle}_{\vec{r}\,'(t)}\,dt$$

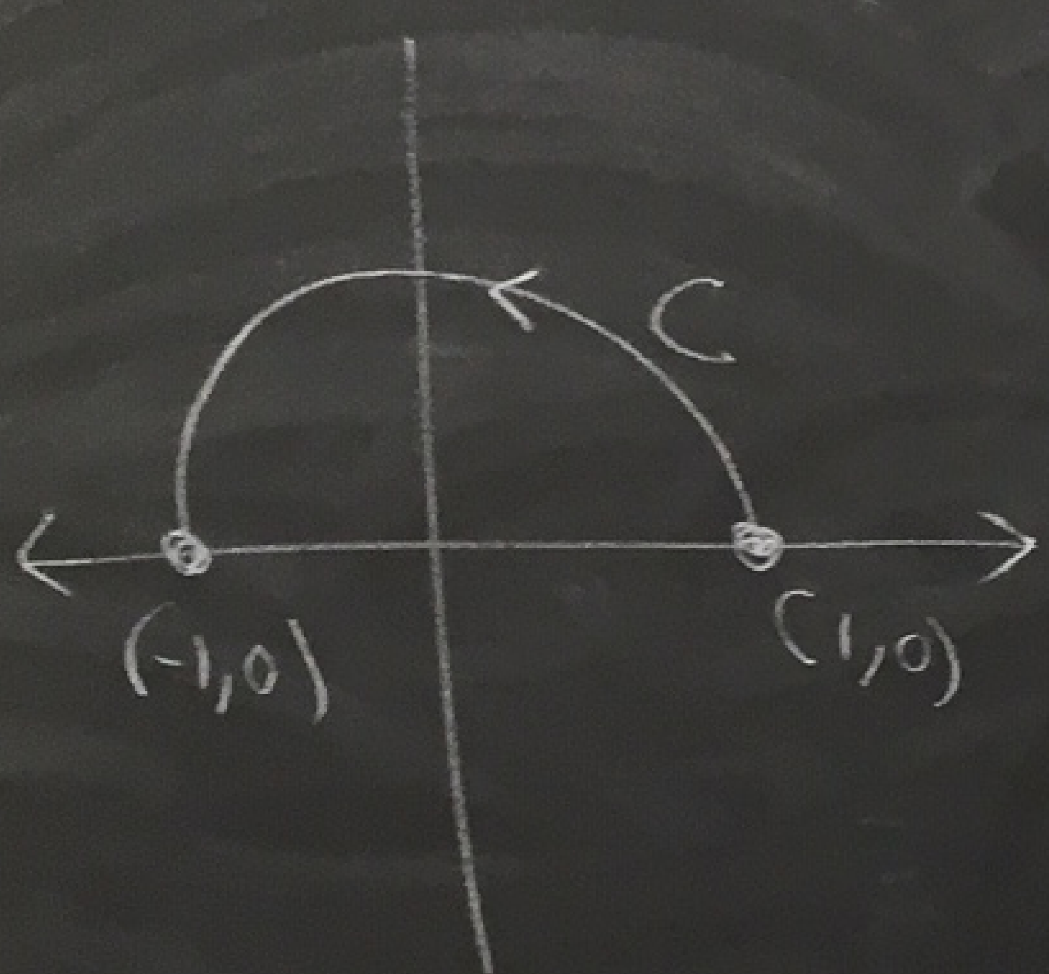
$$\int_a^b f(x(t),y(t))\,\underbrace{x'(t)\,dt}_{dx} + \int_a^b g(x(t),y(t))\,\underbrace{y'(t)\,dt}_{dy}$$

$$= \int_C f\,dx + \int_C g\,dy \quad \leftarrow \boxed{\text{notation}}$$

Ex: Calculate $\int_C \vec{F} \cdot d\vec{r}$ ⟷ $\boxed{\text{work done by } \vec{F} \text{ along } C}$

where $\vec{F}(x,y) = \langle x^2, -xy \rangle$

and C is the top half of the unit circle going counter-clockwise.



$$\vec{r}(t) = \langle \overbrace{\cos(t)}^{x}, \overbrace{\sin(t)}^{y} \rangle, \quad 0 \le t \le \pi$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

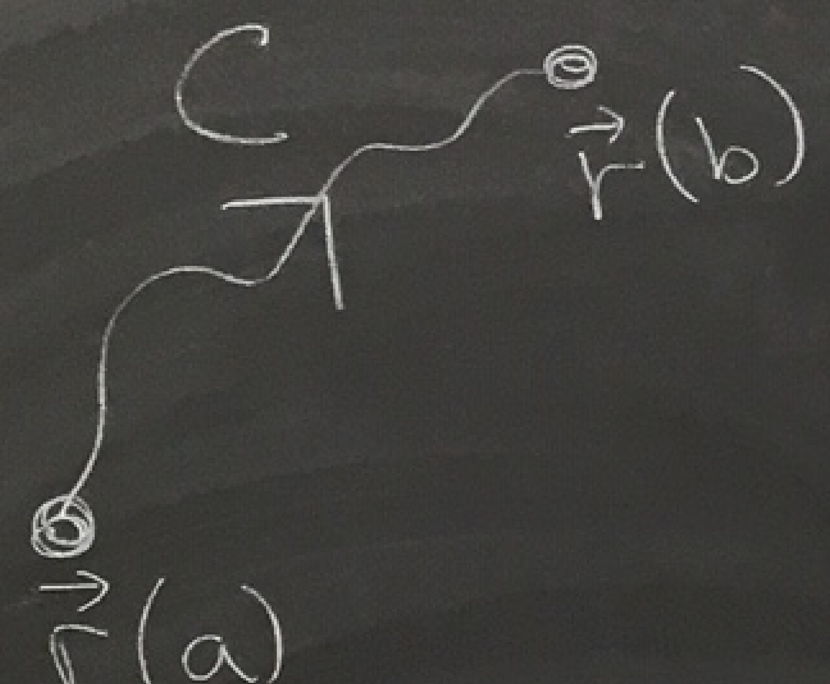
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi} \langle \cos^2(t), -\cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\pi} -\cos^2(t)\sin(t) - \cos^2(t)\sin(t) \, dt = -2\int_0^{\pi} \cos^2(t)\sin(t)\, dt$$

$$= +2\int_1^{-1} u^2 \, du = 2\left.\frac{u^3}{3}\right|_1^{-1} = \frac{2}{3}(-1)^3 - \frac{2}{3}(1)^3 = \boxed{-\frac{4}{3}}$$

$u = \cos(t)$ $du = -\sin(t)\,dt$ $-du = \sin(t)\,dt$	when $t = 0$, $u = \cos(0) = 1$
	when $t = \pi$ $u = \cos(\pi) = -1$



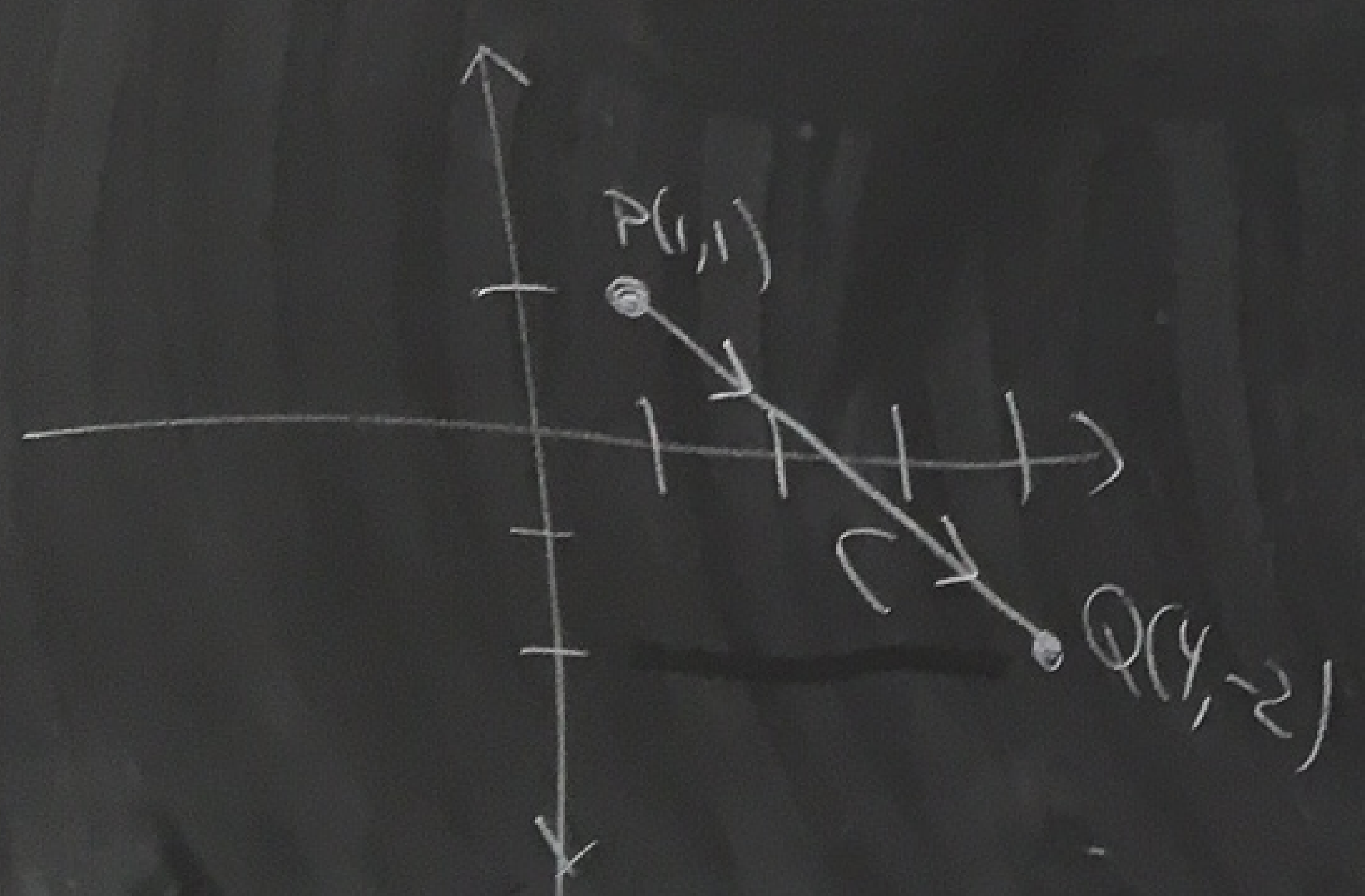
<u>Theorem:</u> Let C be a smooth curve given by a vector function $\vec{r}(t)$ with $a \le t \le b$. [smooth curve means $\vec{r}\,'(t) \neq \vec{0}$ and $\vec{r}\,'(t)$ is continuous]

Suppose $\vec{F}$ is a continuous vector field on C. If there exists a potential function f, that is $\nabla f = \vec{F}$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Ex: Let $\vec{F}(x,y) = \left\langle \frac{y^2}{x^2}, -\frac{2y}{x} \right\rangle$

Calculate $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight-line from $P(1,1)$ to $Q(4,-2)$.



Let's see if we can find $f(x,y)$ with $\nabla f = \vec{F}$

That is, $\langle f_x, f_y \rangle = \left\langle \frac{y^2}{x^2}, -\frac{2y}{x} \right\rangle$

Solve

$$\boxed{\begin{aligned} f_x &= \frac{y^2}{x^2} \quad (1) \\ f_y &= \frac{-2y}{x} \quad (2) \end{aligned}}$$

① $f_x(x,y) = \frac{y^2}{x^2}$

$\int f_x(x,y)\,dx = \int y^2 x^{-2}\,dx$

$f(x,y) = y^2 \frac{x^{-1}}{-1} + \underbrace{C(y)}_{\text{no } x\text{'s}}$

$$\boxed{f(x,y) = -\frac{y^2}{x} + C(y)}$$

take y derivative

$f_y = -\frac{2y}{x} + C'(y)$

plug into ②

$-\frac{2y}{x} + C'(y) = -\frac{2y}{x}$

$C'(y) = 0$

$C(y) = D$

constant

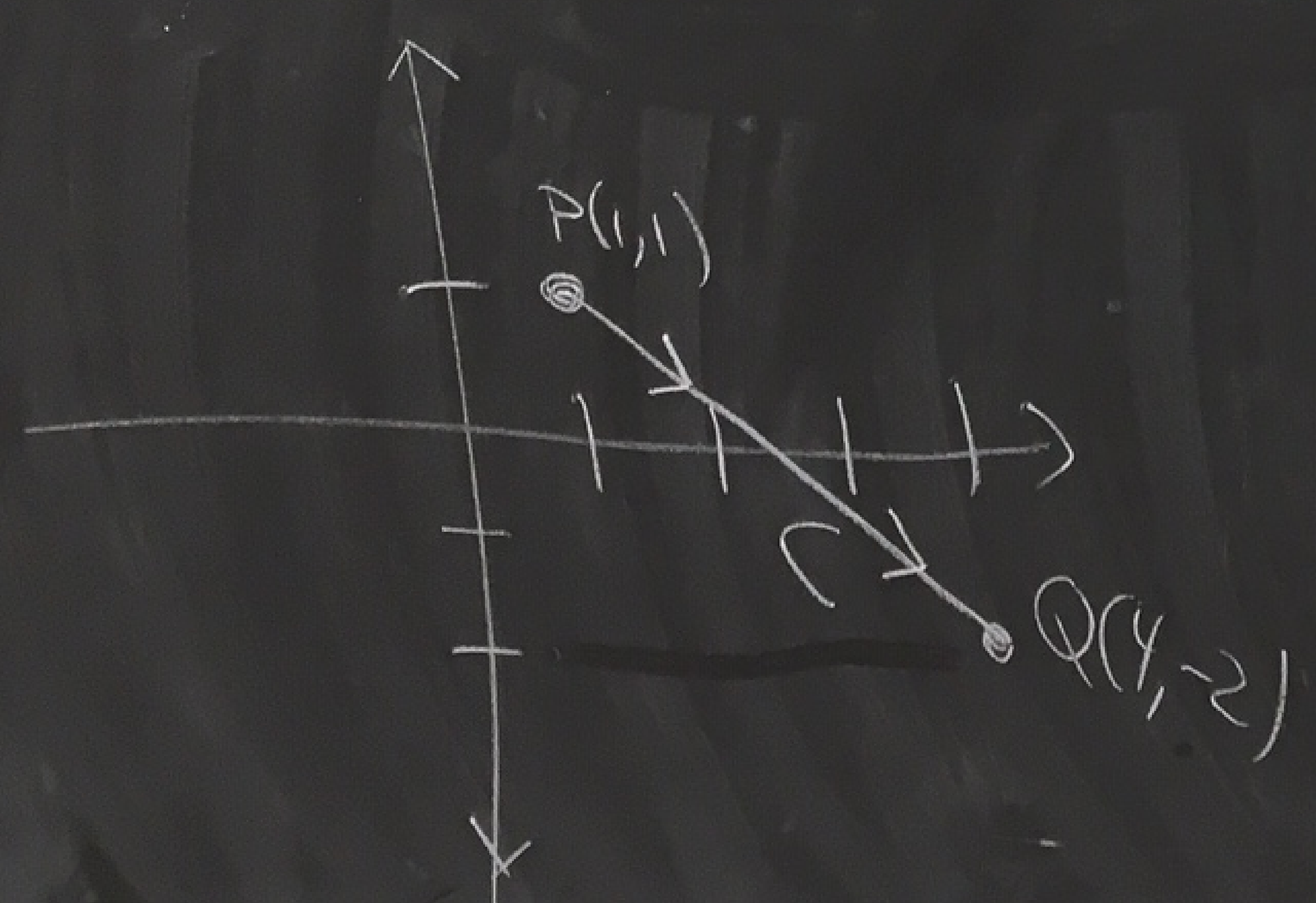
$$\boxed{f(x,y) = -\frac{y^2}{x} + D}$$

Set $D=0$.

So, $f(x,y) = -\dfrac{y^2}{x}$.

Then $\nabla f = \vec{F}$.

So, $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(4,-2) - f(1,1) = \frac{-(-2)^2}{4} - \left(-\frac{1^2}{1}\right) = -1 + 1 = 0.$



14.4 – Green's Theorem

A <u>simple curve</u> is a curve that only intersects itself at its endpoints.

A <u>closed curve</u> is a curve that starts and ends at the same spot.

simple, not closed | not simple, not closed | simple, closed | not simple, closed

Notation:

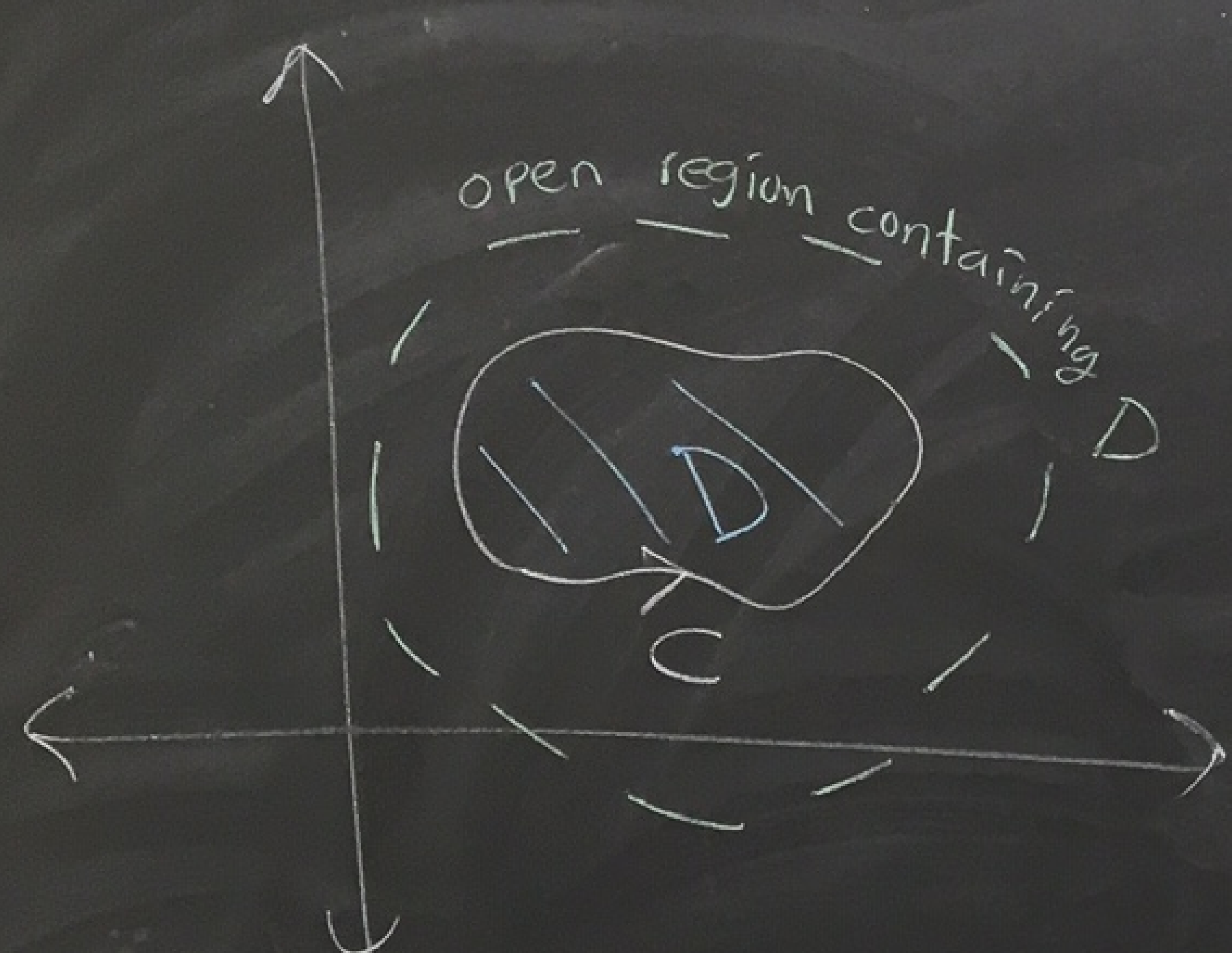
$$\int_C P\,dx + Q\,dy$$

means: If C is given by $\vec{r}(t) = \langle x(t), y(t)\rangle$ with $a \le t \le b$, then

$$\int_C P\,dx + Q\,dy = \int_a^b P(x(t),y(t))\,\underbrace{x'(t)\,dt}_{dx} + \int_a^b Q(x(t),y(t))\,\underbrace{y'(t)\,dt}_{dy} = \int_C \vec{F}\cdot d\vec{r}$$

where $\vec{F} = \langle P, Q\rangle$.

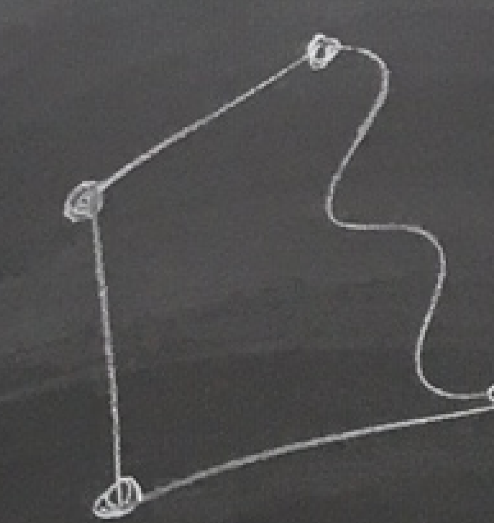
ed



Green's Thm

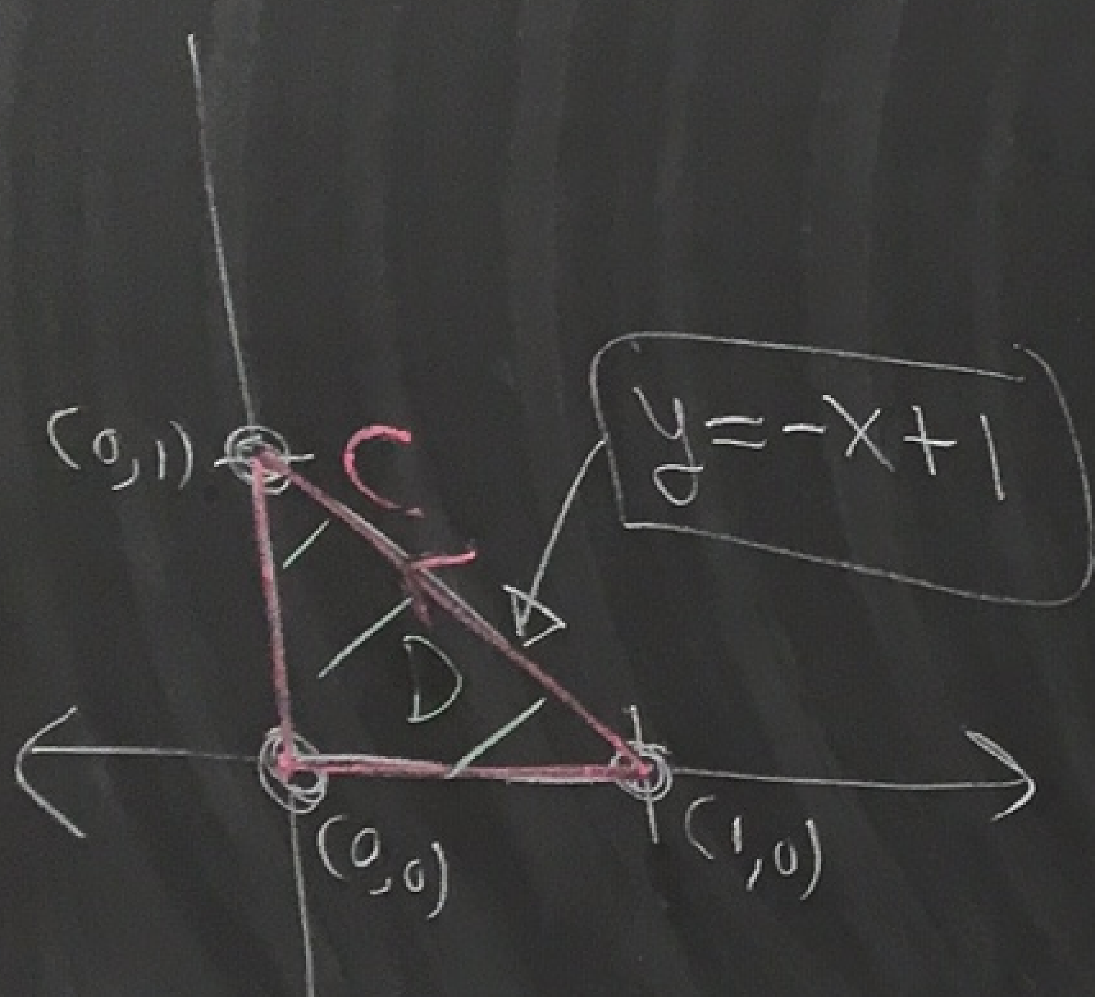
Let C be a piecewise-smooth, simple closed curve oriented in the counterclockwise direction. Let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



Ex: Evaluate $\int_C x^4\,dx + xy\,dy$

where C is the triangular curve with vertices $(0,0)$, $(1,0)$, $(0,1)$.



$$\int_C x^4\,dx + xy\,dy = \iint_D (y-0)\,dA$$

$$\boxed{P(x,y)=x^4} \qquad \boxed{Q(x,y)=xy}$$

$$\boxed{\frac{\partial P}{\partial y}=0} \qquad \boxed{\frac{\partial Q}{\partial x}=y}$$

$$= \int_0^1\int_0^{-x+1} y\,dy\,dx = \int_0^1 \frac{y^2}{2}\Big|_0^{-x+1} dx$$

$$= \frac{1}{2}\int_0^1 (-x+1)^2\,dx = \int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2}\,dx = \frac{1}{6}$$