12/3
Tuesday

4.5 #13

$56 = 2^3 \cdot 7$

If $|G| = 56$, then G has a normal Sylow subgroup.

pf:

We know $n_2 \equiv 1 \pmod{2}$ and $n_2 \mid 7$. So, $n_2 = 1$ or $n_2 = 7$.

We know $n_7 \equiv 1 \pmod{7}$ and $n_7 \mid 8$. So, $n_7 = 1$ or $n_7 = 8$.

We need to show that either $n_2 = 1$ or $n_7 = 1$.

Let's rule out $n_2 = 7$ and $n_7 = 8$.

Suppose $n_2=7$ and $n_7=8$.

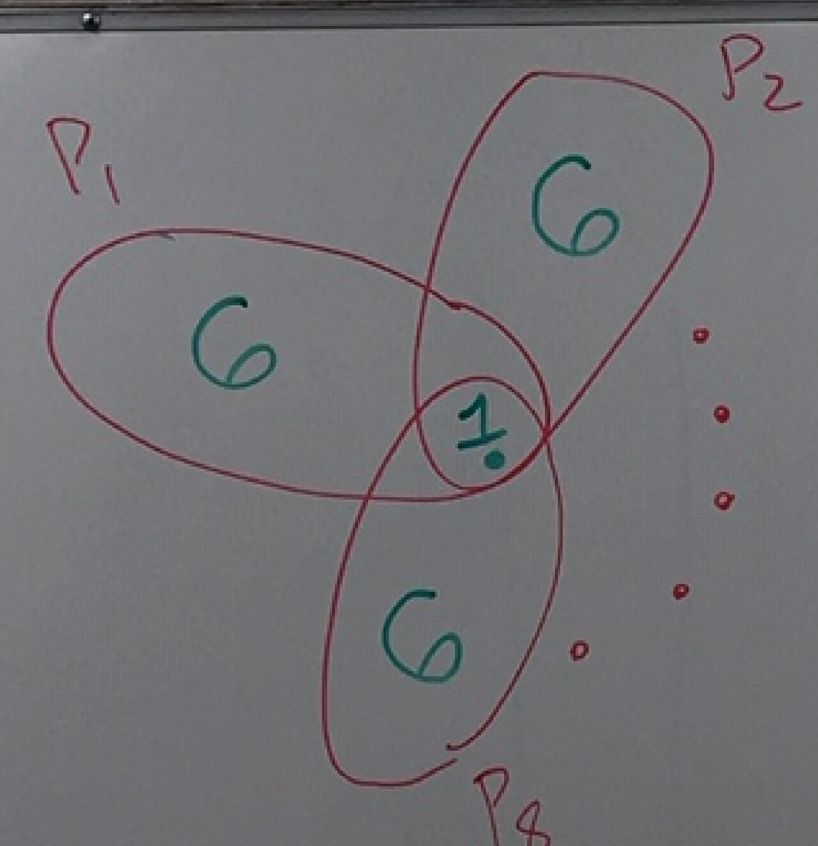
Since $n_7=8$, there are 8 distinct subgroups $P_1, P_2, \dots, P_8$ each of size 7.

Note given $i \neq j$ then $P_i \cap P_j \leq P_i$ so either

$|P_i \cap P_j| = 1$ or $|P_i \cap P_j| = 7$.

If $|P_i \cap P_j| = 7$ then $P_i = P_i \cap P_j = P_j$ which isn't the case.

So, if $i \neq j$ then $P_i \cap P_j = \{1\}$.



So, $|P_1 \cup P_2 \cup \cdots \cup P_8| = 1 + 6 \cdot 8 = 49$

If $n_2 = 7$, there are 7 distinct subgroups of size $8 = 2^3$.

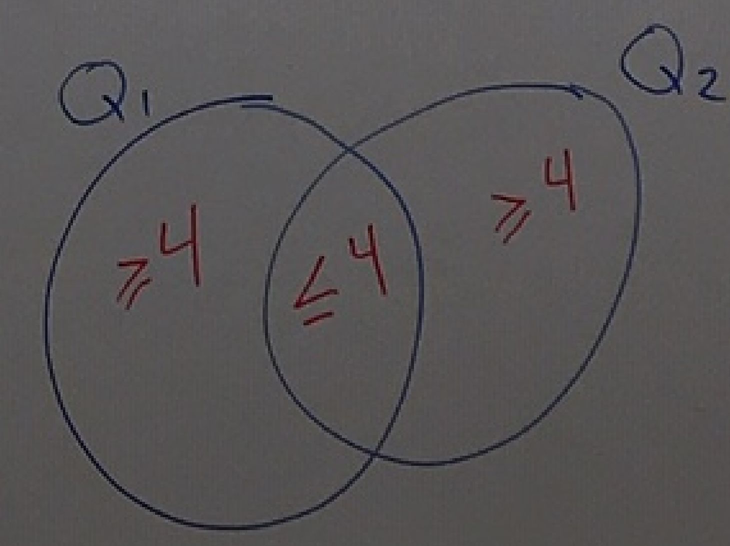
Let Q_1 and Q_2 be two of these.

Then, $|Q_1 \cap Q_2|$ divides 8, so

$|Q_1 \cap Q_2| = 1, 2, 4,$ or 8.

But $|Q_1 \cap Q_2| \neq 8$ since $Q_1 \neq Q_2$.

So, $|Q_1 \cap Q_2| \leq 4$.



So, $Q_1 \cup Q_2$ consists of the identity and at least 8 more elements.

Note that $|P_i \cap Q_j|$ divide $|P_i| = 7$ and $|Q_j| = 8$. So, $|P_i \cap Q_j| = 1$.

But then

$$|P_1 \cup P_2 \cup \cdots \cup P_8 \cup Q_1 \cup Q_2| \geq \underset{\substack{\uparrow \\ \text{identity}}}{1} + \underbrace{6 \cdot 8}_{\text{from } P_i\text{'s}} + \underbrace{2 \cdot 4}_{\text{from } Q_j\text{'s}} = 57 \geq |G|.$$

Contradiction.

So, either $n_2 = 1$ or $n_7 = 1$.

If $n_2 = 1$ we have a normal Sylow 2-subgroup.

If $n_7 = 1$ we have a normal Sylow 7-subgroup.

5.2

A) Let G be a finite abelian group.

Prove G is simple iff $G \cong \mathbb{Z}_p$ where p is a prime.

proof: Let G be a finite abelian group.

($\Longrightarrow$) Suppose G is simple.

So the only normal subgroups of G are $\{1\}$ and G.

Suppose $|G| = n$ where n is not prime.

So, $n = q \cdot m$ where q is a prime and $m > 1$.

By Cauchy there exists $x \in G$ with $|x| = q$.

Let $Q = \langle x \rangle$.

Then $|Q| = |\langle x \rangle| = q$.

$Q \trianglelefteq G$ because G is abelian.

So, $Q \trianglelefteq G$ with $Q \neq \{1\}$ and $Q \neq G$.

This would make G not simple.

So, $|G| = p$ where p is prime.

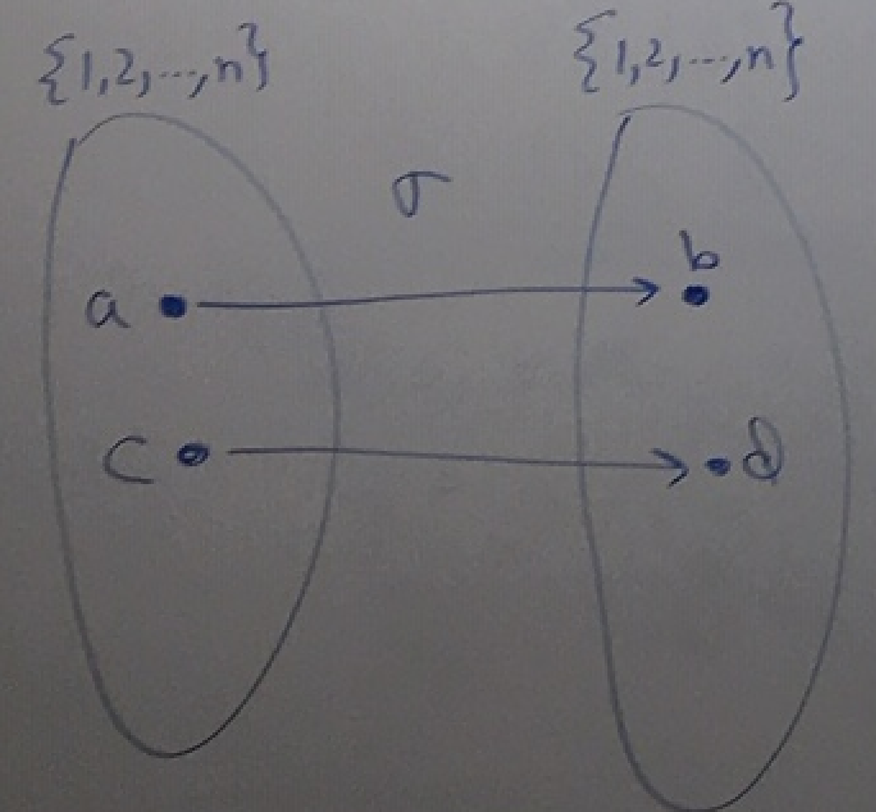
Thus, $G \cong \mathbb{Z}_p$.

($\Longleftarrow$) Suppose $G \cong \mathbb{Z}_p$. So, $|G| = p$ where p is prime.

Let $N \trianglelefteq G$. Then, $|N| = 1$ or $|N| = p$.

So, $N = \{1\}$ or $N = G$.

Thus, G is simple. $\blacksquare$



$a\neq b$
$c\neq d$

4.3 #8 $\quad Z(S_n)=\{1\}$, $n\geq 3$.

Proof: Let $\sigma\in S_n$ with $\sigma\neq 1$.

We will show $\sigma\notin Z(S_n)$.

Since $\sigma\neq 1$, there must exist $a,b,c,d\in\{1,2,3,\dots,n\}$ with $\sigma(a)=b$ and $\sigma(c)=d$ with $a\neq b$ and $c\neq d$.

Note $b\neq d$ since σ is 1-1.

Case 1: $d\neq a$

Let $\tau=(b,d)$.

Then

$$(\sigma\circ\tau)(a)=\sigma(\tau(a))=\sigma(a)=b$$

and

$$(\tau\circ\sigma)(a)=\tau(\sigma(a))=\tau(b)=d$$

So, $\sigma\circ\tau\neq\tau\circ\sigma$.

So, $\sigma\notin Z(S_n)$.

Case 2: $d=a$

Case 2a: $c\neq b$

Let $\tau=(a,b)$.

Then, $(\sigma\circ\tau)(c)=\sigma(\tau(c))=\sigma(c)=a$

and $(\tau\circ\sigma)(c)=\tau(\sigma(c))=\tau(a)=b$

Since $a\neq b$, $\tau\circ\sigma\neq\sigma\circ\tau$.

So, $\sigma\notin Z(S_n)$.

Case 2b: $c=b$

Then $\sigma=(a,b)$.

Let $e\neq a$ and $e\neq b$.

Let $\tau=(b,e)$.

Then,

$$(\sigma\circ\tau)(b)=\sigma(\tau(b))=\sigma(e)=e$$

and $(\tau\circ\sigma)(b)=\tau(\sigma(b))=\tau(a)=a$

So, $\sigma\circ\tau\neq\tau\circ\sigma$.

So, $\sigma\notin Z(S_n)$. ∎