Math 2120 - Test 1

Name: Solutions

Directions: Show all of your work to get credit. No calculators. Good luck!

Score	
1	
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1. Calculate the following integral.

$$\int t^2 e^{-t} dt$$

$$\int t^2 e^{-t} dt = -t^2 e^{-t} + \int 2t e^{-t} dt$$

$$\boxed{\begin{array}{ll} u = t^2 & du = 2t\,dt \\ dv = e^{-t} dt & v = -e^{-t} \end{array}}$$

$$= -t^2 e^{-t} + 2\int t e^{-t} dt$$

$$= -t^2 e^{-t} + 2\left[-t e^{-t} + \int e^{-t} dt\right]$$

$$\boxed{\begin{array}{ll} u = t & du = dt \\ dv = e^{-t} dt & v = -e^{-t} \end{array}}$$

$$= \boxed{-t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + C}$$

2. Calculate the following integral.

$$\int \sin^4(x)\cos^5(x)dx$$

$$\int \sin^4(x)\cos^5(x)dx$$

$$= \int \sin^4(x)\cos^4(x)\cos(x)dx$$

$$= \int \sin^4(x)\left[\cos^2(x)\right]^2\cos(x)dx$$

$$= \int \sin^4(x)\left[1-\sin^2(x)\right]^2\cos(x)\,dx$$

$$= \int u^4\left[1-u^2\right]^2du = \int u^4\left(1-2u^2+u^4\right)du$$

$$\boxed{\begin{array}{l} u=\sin(x) \\ du=\cos(x)dx \end{array}}$$

$$= \int\left(u^4-2u^6+u^8\right)du$$

$$= \frac{u^5}{5} - \frac{2}{7}u^7 + \frac{u^9}{9} + C$$

$$= \boxed{\frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9(x) + C}$$

3. Calculate the following integral.

$$\int \frac{-2x^2+6}{x^3-x}\,dx$$

$$\frac{-2x^2+6}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$-2x^2+6 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$\underline{x=1;}$ $-2(1)^2+6 = A(0)+B(2)+C(0)$

$B=2$

$\underline{x=-1;}$ $-2(-1)^2+6 = A(0)+B(0)+C(-1)(-2)$

$C=2$

$\underline{x=0:}$ $6 = A(-1)(1) \rightarrow A=-6$

$$\int \frac{-2x^2+6}{x^3-x}\,dx = \int \left[\frac{-6}{x} + \frac{2}{x-1} + \frac{2}{x+1}\right] dx$$

$$= -6\ln|x| + 2\ln|x-1| + 2\ln|x+1| + C$$

4. Does the following integral converge or diverge? If it converges, what does it converge to?

$$\int_0^\infty xe^{-x^2}\,dx$$

$$\int xe^{-x^2}dx = \int \left(-\tfrac{1}{2}e^u\right) du = -\tfrac{1}{2}e^u + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$\boxed{\begin{array}{c} u = -x^2 \\ du = -2x\,dx \\ -\frac{1}{2}du = x\,dx \end{array}}$$

So,

$$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty} \int_0^t xe^{-x^2}dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-x^2} \Big|_0^t \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}e^0 \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2e^{t^2}} + \frac{1}{2} \right] = 0 + \frac{1}{2} = \frac{1}{2}$$

So the integral converges to $1/2$

5. Does the following integral converge or diverge? If it converges, what does it converge to?

$$\int_3^5 \frac{dx}{\sqrt{x-3}}$$

$$\int_3^5 \frac{dx}{\sqrt{x-3}} = \lim_{t\to 3^+} \int_t^5 \frac{dx}{\sqrt{x-3}}$$

$$= \lim_{t\to 3^+} \int_t^5 (x-3)^{-1/2}\,dx$$

$$= \lim_{t\to 3^+} \left. \frac{(x-3)^{1/2}}{1/2} \right|_t^5$$

$$= \lim_{t\to 3^+} \left[2\sqrt{x-3} \right]_t^5$$

$$= \lim_{t\to 3^+} \left[2\sqrt{5-3} - 2\sqrt{t-3} \right]$$

$$= 2\sqrt{2} - 2\sqrt{3-3} = 2\sqrt{2} - 0$$

$$= 2\sqrt{2}$$

6. Calculate the following integral.

$$\int_{3/2}^{3} \sqrt{9 - x^2}\, dx$$

$$\int_{3/2}^{3} \sqrt{9-x^2}\, dx \underset{\uparrow}{=} \int_{\pi/3}^{\pi/2} \sqrt{9-9\sin^2(\theta)} \cdot 3\cos(\theta) d\theta$$

$$\boxed{\begin{array}{l} x = 3\sin(\theta) \\ dx = 3\cos(\theta)d\theta \\ x = \frac{3}{2} \to \frac{3}{2} = 3\sin(\theta) \to \frac{1}{2} = \sin(\theta) \to \theta = \frac{\pi}{6} \\ x = 3 \to 3 = 3\sin(\theta) \to 1 = \sin(\theta) \to \theta = \frac{\pi}{2} \end{array}}$$

$$= 3\int_{\pi/6}^{\pi/2} \sqrt{9}\sqrt{1-\sin^2(\theta)} \cdot \cos(\theta) d\theta$$

$$= 9\int_{\pi/6}^{\pi/2} \sqrt{\cos^2(\theta)}\, \cos(\theta)\, d\theta$$

$$= 9\int_{\pi/6}^{\pi/2} \cos^2(\theta) d\theta = 9\int_{\pi/6}^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$$

$$= 9\left[\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right]_{\pi/6}^{\pi/2}$$

$$= 9\left[\left(\frac{\pi/2}{2} + \frac{1}{4}\sin\left(\frac{2\pi}{2}\right)\right) - \left(\frac{\pi/6}{2} + \frac{1}{4}\sin\left(\frac{2\pi}{6}\right)\right)\right]$$

$$= 9\left[\frac{\pi}{4} + \underbrace{\frac{1}{4}\sin(\pi)}_{0} - \frac{\pi}{12} - \frac{1}{4}\underbrace{\sin\left(\frac{2\pi}{6}\right)}_{\sqrt{3}/2}\right]$$

$$= \frac{9\pi}{4} - \frac{9\pi}{12} - \frac{9}{4}\frac{\sqrt{3}}{2}$$

6. Calculate the following integral.

$$\int_{3/2}^{3} \sqrt{9-x^2}\, dx$$

$$= \frac{18\pi}{12} - \frac{9\sqrt{3}}{8}$$

$$= \frac{3\pi}{2} - \frac{9\sqrt{3}}{8}$$

$$= \frac{12\pi - 9\sqrt{3}}{8}$$