Test 2 Solutions

(1) $$\lim_{n\to\infty} \frac{n^{10}-5n^{8}-1}{2n^{15}+4n^{29}+5} = \lim_{n\to\infty} \frac{\frac{1}{n^{19}}-5\frac{1}{n^{21}}-\frac{1}{n^{29}}}{2\frac{1}{n^{14}}+4+\frac{5}{n^{29}}}$$

$$= \frac{0-0-0}{0+4+0} = \frac{0}{4}$$

$$= 0$$

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(2) $$\sum_{k=2}^{\infty} 5\frac{\pi^k}{10^{k+1}} = \frac{5}{10}\sum_{k=2}^{\infty}\left(\frac{\pi}{10}\right)^k$$

$$= \frac{5}{10}\left[\left(\frac{\pi}{10}\right)^2+\left(\frac{\pi}{10}\right)^3+\left(\frac{\pi}{10}\right)^4+\cdots\right]$$

$$= \frac{5}{10}\left(\frac{\pi}{10}\right)^2\left[1+\frac{\pi}{10}+\left(\frac{\pi}{10}\right)^2+\cdots\right]$$

$\pi \approx 3.14\ldots$

$0 < \frac{\pi}{10} < 1$

$$= \frac{5}{10}\left(\frac{\pi}{10}\right)^2\left[\frac{1}{1-\frac{\pi}{10}}\right]$$

$$= \frac{5}{10}\,\frac{\pi^2}{10^2}\cdot\frac{1}{\frac{10-\pi}{10}} = \frac{5\pi^2}{10^3}\cdot\frac{10}{10-\pi} = \frac{5\pi^2}{10^2(10-\pi)}$$

(3) $$\frac{1}{(k+9)(k+8)} = \frac{A}{k+9} + \frac{B}{k+8}$$

$$1 = A(k+8) + B(k+9)$$

$\underline{k=-8}$: $1 = A(0) + B(-8+9) \leftarrow 1 = B(1)$, $B = 1$

$\underline{k=9}$: $1 = A(-9+8) + B(0) \leftarrow A = -1$

$$\sum_{k=4}^{\infty} \frac{1}{(k+9)(k+8)} = \sum_{k=4}^{\infty} \left[\frac{1}{k+8} - \frac{1}{k+9}\right]$$

$$= \left(\frac{1}{12} - \frac{1}{13}\right) + \left(\frac{1}{13} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{15}\right) + \dots$$

$$S_1 = \frac{1}{12} - \frac{1}{13}$$

$$S_2 = \left(\frac{1}{12} - \frac{1}{13}\right) + \left(\frac{1}{13} - \frac{1}{14}\right) = \frac{1}{12} - \frac{1}{14}$$

$$S_3 = \left(\frac{1}{12} - \frac{1}{13}\right) + \left(\frac{1}{13} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{15}\right) = \frac{1}{12} - \frac{1}{15}$$

$\vdots \quad \vdots$

$$S_n = \frac{1}{12} - \frac{1}{n+12}$$

$$\lim_{n \to \infty} S_n = \frac{1}{12}$$

Note for #3   Common mistake   pg 2'

You can't just write

$$\sum_{k=4}^{\infty} \frac{1}{(k+9)(k+8)} = \sum_{k=4}^{\infty} \left(\frac{1}{k+8} - \frac{1}{k+9} \right)$$

$$= \left(\frac{1}{12} - \cancel{\frac{1}{13}} \right) + \left(\cancel{\frac{1}{13}} - \cancel{\frac{1}{14}} \right) + \left(\cancel{\frac{1}{14}} - \cancel{\frac{1}{15}} \right) + \dots$$

$$= \frac{1}{12}$$

Why?

Ex: Consider $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty} \left[\ln(k+1) - \ln(k)\right]$

It appears that we get

$$\left[\cancel{\ln(2)} - \ln(1)\right] + \left[\cancel{\ln(3)} - \cancel{\ln(2)}\right] + \left[\cancel{\ln(4)} - \cancel{\ln(3)}\right] + \dots$$

$$= -\ln(1) = 0$$

But this is incorrect.

Why?

The sum really behaves this way!

$$S_1 = \left(\ln(2) - \ln(1)\right) = \ln(2)$$

$$S_2 = \left(\cancel{\ln(2)} - \underbrace{\ln(1)}_{0}\right) + \left(\ln(3) - \cancel{\ln(2)}\right) = \ln(3)$$

$$S_3 = \left(\cancel{\ln(2)} - \underbrace{\ln(1)}_{0}\right) + \left(\cancel{\ln(3)} - \cancel{\ln(2)}\right) + \left(\ln(4) - \cancel{\ln(3)}\right)$$
$$= \ln(4)$$

$\vdots$

So, $S_n = \ln(n+1)$

That is,

$$\sum_{k=1}^{\infty}\left[\ln(k+1) - \ln(k)\right] = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1)$$
$$= \infty$$

This sum actually diverges.

(4) $$\lim_{k\to\infty} \frac{e^k}{e^k+10} = \lim_{k\to\infty} \frac{1}{1+\frac{10}{e^k}}$$

$$= \frac{1}{1+0} = 1 \neq 0$$

So, $\sum \frac{e^k}{e^k+10}$ does not converge by the divergence test.

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(5) $$\sum_{k=2}^{\infty} \frac{2}{k\ln(k)}$$

Integral test time

$$f(x) = \frac{2}{x\ln(x)}$$

- $f(x) = \frac{2}{x\ln(x)} > 0$ for $x \geq 2$.
- $f(x)$ is continuous for $x \geq 2$.

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- $f'(x) = \left(\frac{2}{x\ln(x)}\right)' = \frac{0(x\ln(x)) - 2(1\cdot\ln(x) + x\cdot\frac{1}{x})}{(x\ln(x))^2}$

$$= \frac{-2-2\ln(x)}{(x\ln(x))^2} = \frac{-2(1+\ln(x))}{(x\ln(x))^2} < 0$$ when $x \geq 2$

~~Note that ln(x) when x ≥ 2~~

So, $f'(x) = \frac{-2(1+\ln(x))}{(x\ln(x))^2} < 0$ when $x \geq 2$

So, f is decreasing when $x \geq 2$ ~~since then~~

~~when~~

So we can apply the integral test.

$$\int_2^\infty \frac{2}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_2^t \frac{2}{x\ln(x)}\,dx = \lim_{t\to\infty} 2\ln(\ln(x))\Big|_2^t$$

$$\boxed{\begin{aligned} 2\int \frac{dx}{x\ln(x)} &= 2\int \frac{1}{u}\,du \\ u=\ln(x) \quad &= 2\ln|u| + C \\ du = \tfrac{1}{x}dx \quad &= 2\ln(\ln(x)) + C \end{aligned}}$$

$$= \lim_{t\to\infty}\left[\underbrace{2\ln(\ln(t))}_{\infty} - 2\ln(\ln(2))\right] = \infty$$

So the series diverges

⑥ $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2+1}$

$a_k = \frac{5}{k^2+1}$ Alternating series test

- $a_k > 0$ ✓

- $a_k = \frac{5}{k^2+1}$

  $a_{k+1} = \frac{5}{(k+1)^2+1}$

  Since $(k+1)^2 > k^2$ we get $\frac{5}{(k+1)^2+1} < \frac{5}{k^2+1}$

  So, $a_{k+1} < a_k$.

- $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{5}{k^2+1} = 0$

So by the alternating series test

$\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2+1}$ converges.

7 Let's use the ratio test.

$$L = \lim_{k\to\infty} \left| \frac{\frac{(k+1)\cdot 10^{k+1}}{(k+1)!}}{\frac{k\cdot 10^k}{k!}} \right|$$

$$= \lim_{k\to\infty} \frac{(k+1)\cdot 10^{k+1}}{(k+1)!} \cdot \frac{k!}{k\cdot 10^k}$$

$$= \lim_{k\to\infty} \frac{k+1}{k} \cdot \frac{k!}{(k+1)!} \cdot \frac{10^{k+1}}{10^k}$$

$$= \lim_{k\to\infty} \frac{k+1}{k} \cdot \frac{k!}{(k+1)\cdot[k!]} \cdot \frac{10}{1}$$

$$= \lim_{k\to\infty} \frac{k+1}{k} \cdot \frac{1}{k+1} \cdot 10$$

$$= \lim_{k\to\infty} \frac{10}{k} = 0$$

Since $0 \le L < 1$, the series converges.