Math 2150-01
2/10/25

I put a study guide with practice tests on the website.

Test 1 is March 17

Topic 5 – First order exact equations

Suppose you have a first order equation of the form:

$$\underbrace{M(x,y)} + \underbrace{N(x,y)} \cdot y' = 0$$

expressions with x and y

Ex:

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}\, y' = 0$$

Suppose also that there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

Ex: $\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$

Let $f(x,y) = x^2 y - y$

Then,

$$\frac{\partial f}{\partial x} = 2xy + 0 = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Suppose $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

Then,

$$M(x,y) + N(x,y) \cdot y' = 0$$

becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

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$f(x,y)$ is a function of x and y

$y = y(x)$ is a function of x

chain rule:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} \cdot \frac{d}{dx}(y)$$

$$= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

Hence, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

becomes $\frac{df}{dx} = 0$

So, for example the family of curves given by $f(x,y) = c$ where c is a constant will satisfy $\frac{df}{dx} = 0$.

<u>Summary</u>: If $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$, then the equation $f(x,y) = c$ where c is any constant will give an implicit solution to

$$M(x,y) + N(x,y) \cdot y' = 0$$

If such an f exists then we say that

$$M(x,y) + N(x,y)\cdot y' = 0$$

is an exact equation

Ex: Consider

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

Let

$$f(x,y) = x^2 y - y$$

We have

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Thus, $x^2y - y = c$ ← $f(x,y)=c$

Where c is any constant is an implicit solution to $2xy + (x^2-1)y' = 0$

$\boxed{\text{Check \#1}}$

Suppose $x^2y - y = c$.
Differentiate both sides with respect to x to get:

$$2xy + x^2\underbrace{y'}_{\frac{dy}{dx}} - \underbrace{y'}_{\frac{dy}{dx}} = 0$$

So,

$$2xy + (x^2-1)y' = 0$$

original equation

$\boxed{\text{Check \#2}}$

We can actually solve for y in our solution $x^2y - y = c$.

We get

$$y = \frac{c}{x^2-1}$$

Let's check if it solves the equation. We have

$$y = c(x^2-1)^{-1}$$

$$y' = -c(x^2-1)^{-2}\cdot(2x)$$

$$= \frac{-2cx}{(x^2-1)^2}$$

Plug this into the ODE to get

$$2xy + (x^2-1)y'$$

$$= 2x\underbrace{\left(\frac{c}{x^2-1}\right)}_{y} + (x^2-1)\underbrace{\left(\frac{-2cx}{(x^2-1)^2}\right)}_{y'}$$

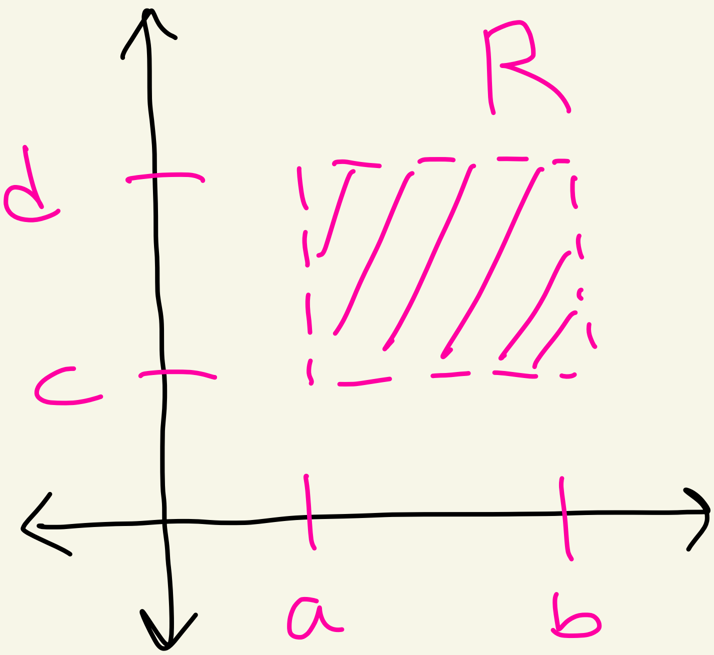
$$= \frac{2xc}{x^2-1} + \frac{-2cx}{x^2-1} = 0$$

Thus, $y = \dfrac{c}{x^2-1}$ solves

$$2xy + (x^2-1)y' = 0$$

When does such an f exist?

Theorem Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<x<d$.



Then,

$M(x,y)+N(x,y)y'=0$

will be exact if and only if

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Here a,b,c,d can be $\pm\infty$

Proof: See notes if interested.

Ex: Consider the previous equation

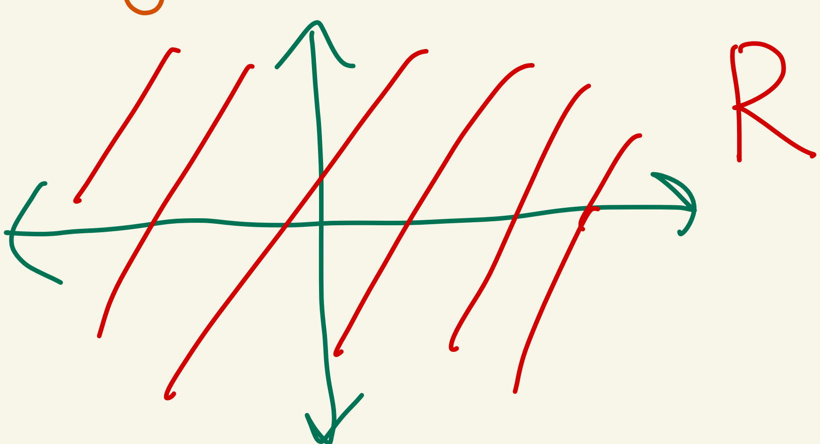
$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}\, y' = 0$$

We have

$$\left.\begin{aligned} M(x,y) &= 2xy \\ N(x,y) &= x^2-1 \end{aligned}\right\} \text{these are continuous everywhere}$$

And

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y & \qquad \frac{\partial N}{\partial x} &= 2x \\ \frac{\partial M}{\partial y} &= 2x & \qquad \frac{\partial N}{\partial y} &= 0 \end{aligned}\right\} \text{these are continuous everywhere}$$



R is the entire xy-plane

And,

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

Equal!

Thus, $2xy + (x^2-1)y' = 0$ is exact, that is there exists $f(x,y)$ where $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$