Math 2150-01
2/12/25

(topic 5 continued...)

Last time we saw that

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

is exact. We saw that a solution is given by $x^2y - y = c$ where $f(x,y) = x^2y - y$.

How did I find such an f? Pretend like we don't know f. We need f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2 - 1 \quad (2) \end{aligned}}$$

$$\leftarrow \boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M(x,y) \\ \frac{\partial f}{\partial y} &= N(x,y) \end{aligned}}$$

Start with 1:

$$\frac{\partial f}{\partial x} = 2xy$$

Integrate with respect to x:

$$f(x,y) = x^2 y + \underbrace{C(y)}_{\text{constant with \#'s and y's but no x's}}$$

We want to plug this into 2. First take the y derivative. We get

$$\frac{\partial f}{\partial y} = x^2 + C'(y)$$

Plug this into the left side of 2 and get:

$$x^2 + c'(y) = x^2 - 1$$

② $\frac{\partial f}{\partial y} = x^2 - 1$

So,

$$c'(y) = -1$$

Thus,

$$c(y) = -y + D$$

where D is a constant.

Thus,

$$f(x,y) = x^2 y + c(y)$$

$$= x^2 y - y + D$$

You can just set $D = 0$ because the solution is $f(x,y) = c$

So it would be $x^2 y - y + D = c$

So, solution is $x^2 y - y = \underbrace{c - D}_{\text{another constant}}$

So use

$$\boxed{f(x,y) = x^2y - y}$$

HW 5 | 2(b)

Consider the initial value problem

$$(e^x + y) + (2 + x + ye^y)y' = 0$$
$$y(0) = 1$$

Let's first solve

$$\underbrace{(e^x + y)}_{M(x,y)} + \underbrace{(2 + x + ye^y)}_{N(x,y)}y' = 0$$

We have

$$M(x,y) = e^x + y$$
$$N(x,y) = 2 + x + ye^y$$

these are continuous everywhere

$$\frac{\partial M}{\partial x} = e^x \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial y} = 1 \cdot e^y + y \cdot e^y$$

<u>Check:</u> $\frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}$

So, the ODE equation is exact.

Thus, the equation

$$(e^x + y) + (2 + x + ye^y)\, y' = 0$$

is exact. So there must exist

$f(x,y)$ that satisfies:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

$\frac{\partial f}{\partial x} = M(x,y)$

$\frac{\partial f}{\partial y} = N(x,y)$

Start with (1):

$$\frac{\partial f}{\partial x} = e^x + y$$

Integrate with respect to x:

$$f(x,y) = e^x + yx + \underbrace{C(y)}_{\substack{\text{constant with} \\ \#\text{'s and } y\text{'s} \\ \text{no } x\text{'s}}}$$

Take the y derivative:

$$\frac{\partial f}{\partial y} = x + C'(y)$$

Plug this into the left side of (2):

$$x + c'(y) = 2 + x + ye^y$$

So,

$$c'(y) = 2 + ye^y$$

Thus,

$$c(y) = \int (2 + ye^y)\,dy$$

$$= 2y + \underbrace{\int ye^y\,dy}$$

LIATE $\longrightarrow$

$\int u\,dv = uv - \int v\,du$

$u = y \qquad du = dy$

$dv = e^y dy \qquad v = e^y$

$$= 2y + \left(ye^y - \int e^y\,dy \right)$$

$$= 2y + ye^y - e^y$$

(We don't need + constant here since we set $f(x,y) = $ constant in answer)

So,

$$f(x,y) = e^x + yx + c(y)$$

$$= e^x + yx + \underbrace{2y + ye^y - e^y}_{c(y)}$$

$$\boxed{\begin{array}{l}\text{A solution to} \\ \quad (e^x + y) + (2 + x + ye^y)y' = 0 \\ \text{is} \\ \quad e^x + yx + 2y + ye^y - e^y = c \\ \text{where } c \text{ is a constant.}\end{array}}$$

$$\boxed{f(x,y) = c}$$

Now let's find a solution where $y(0) = 1$.

Plug $x = 0$, $y = 1$ into our solution.

We get:

$$\underbrace{e^0} + \underbrace{(1)(0)} + \underbrace{2(1)} + \underbrace{(1)e^1 - e^1} = c$$

1 0 2 0

So, $c = 3$.

Answer to initial value problem is:

$$\boxed{e^x + yx + 2y + ye^y - e^y = 3}$$

Method 2 to solve:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \qquad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \qquad (2) \end{aligned}}$$

(1): $f(x,y) = e^x + yx + C(y)$

(2): $f(x,y) = 2y + xy + ye^y - e^y + D(x)$

Set equal:

$$e^x + \cancel{yx} + C(y) = 2y + \cancel{xy} + ye^y - e^y + D(x)$$

Simplify:

$$e^x + C(y) = 2y + ye^y - e^y + D(x)$$

Plug $C(y)$ into ① eqn:

$$f(x,y) = e^x + yx + C(y)$$

$$= e^x + yx + 2y + ye^y - e^y$$