Math 2150-01
2/19/25

Topic 6 continued...

For the remainder of topic 6 we will be learning the theory of solving 2nd order linear ODE

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)}$$

on some interval I where $a_2(x), a_1(x), a_0(x), b(x)$ are continuous on I and $a_2(x) \neq 0$ on I. We will assume these conditions for the rest of topic 6.

Ex: $$x^2 y'' \underbrace{-4x}_{} y' + 6y = \frac{1}{x}$$

$a_2(x) = x^2$ $a_1(x) = -4x$ $a_0(x) = 6$ $b(x) = \frac{1}{x}$

$I = (0, \infty)$

$\boxed{\text{Fact 1:}}$ If $f_1(x)$ and $f_2(x)$ are linearly independent solutions to the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

on I, then every solution to $(*)$ on I is of the form

$$\boxed{y_h = c_1 f_1(x) + c_2 f_2(x)}$$

where c_1, c_2 are constants.

$\boxed{\text{Fact 2:}}$ Suppose we can find a particular solution y_p to

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)} \quad (**)$$

on I, then every solution to $(**)$ on I is of the form

$$y = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{y_h} + y_p$$

homogeneous solution

Ex: Let's solve

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$

Step 1: Solve the homogeneous equation:

$$y'' - 7y' + 10y = 0$$

Consider $f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.
Last time we showed that f_1 and f_2 are linearly independent. Let's show they both solve $\boxed{y'' - 7y' + 10y = 0}$.
Let's plug them in.

We have:

$f_1 = e^{2x}$, $f_1' = 2e^{2x}$, $f_1'' = 4e^{2x}$

$f_2 = e^{5x}$, $f_2' = 5e^{5x}$, $f_2'' = 25e^{5x}$

Plug them in to get:

$$
\begin{aligned}
f_1'' - 7f_1' + 10f_1 &= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x}) \\
&= (4 - 14 + 10)e^{2x}
\end{aligned}
$$

$$= 0$$

And,

$$f_2'' - 7f_2' + 10f_2$$

$$= 25e^{25} - 7(5e^{5x}) + 10(e^{5x})$$

$$= (25 - 35 + 10)e^{5x}$$

$$= 0$$

<u>Summary</u>: Since f_1 and f_2 are linearly independent solutions to $y'' - 7y' + 10y = 0$ that means that <u>all</u> solutions to $y'' - 7y' + 10y = 0$ are of the form

$$y_h = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{c_1 f_1 + c_2 f_2}$$

where c_1, c_2 are any constants

Example solutions are

$$y_h = 10e^{2x} - 1000e^{5x}$$

$$y_h = \frac{1}{2}e^{2x} + 0e^{5x} = \frac{1}{2}e^{2x}$$

Step 2: Let's now solve

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$.

Consider

$$y_p = 6e^{x}$$

We will learn how to find this later

Let's verify that y_p solves

$$y'' - 7y' + 10y = 24e^{x}$$

We get

$$y_p = 6e^x, \quad y_p' = 6e^x, \quad y_p'' = 6e^x$$

So,

$$y_p'' - 7y_p' + 10y_p$$
$$= 6e^x - 7(6e^x) + 10(6e^x)$$
$$= (6 - 42 + 60)e^x$$
$$= 24e^x$$

<u>Answer:</u> Every solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$, is of the form

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{\substack{\text{general solution } y_h \\ \text{to the homogeneous} \\ y'' - 7y' + 10y = 0}} + \underbrace{6e^x}_{\substack{\text{particular} \\ \text{solution } y_p \text{ to} \\ y'' - 7y' + 10y = 24e^x}}$$

Ex: Let's find all the solutions to

$$\boxed{x^2 y'' - 4xy' + 6y = \frac{1}{x}}$$

on $I = (0, \infty)$

Step 1: First solve the homogeneous equation

$$x^2 y'' - 4xy' + 6y = 0$$

Consider

$f_1(x) = x^2$

$f_2(x) = x^3$

We will see how to find these later

First we check that f_1, f_2 are linearly independent.

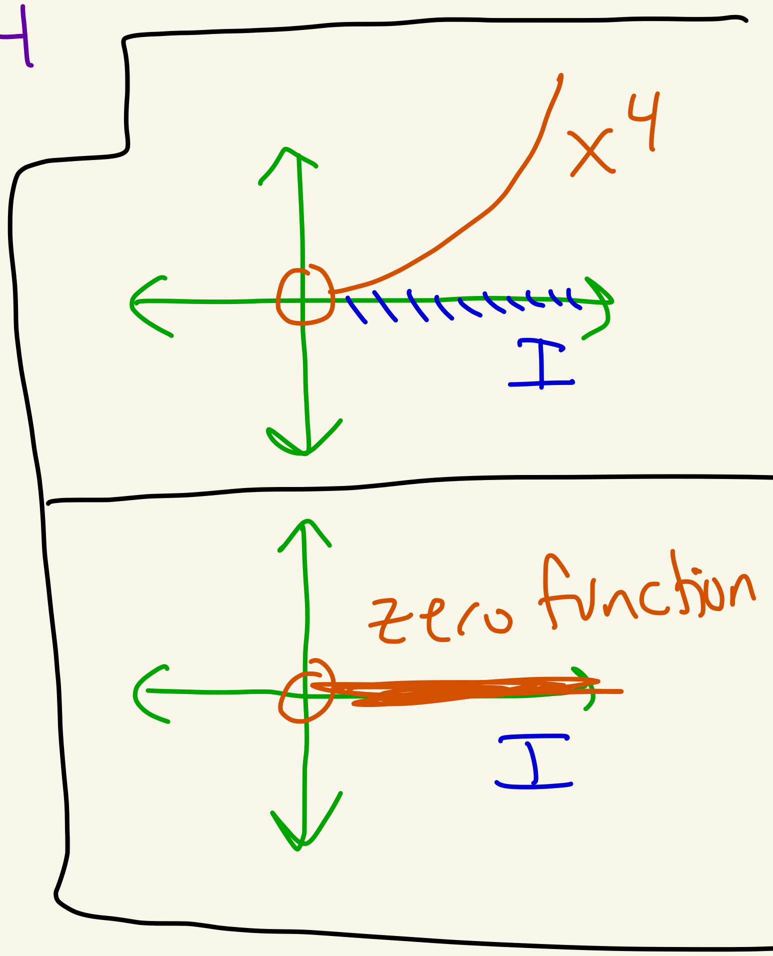
We have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= (x^2)(3x^2) - (2x)(x^3)$$

$$= x^4$$



This is not the zero function on $I = (0, \infty)$.

So, f_1, f_2 are linearly independent on $I = (0, \infty)$

Now we check that f_1, f_2 solve $x^2y'' - 4xy' + 6y = 0$.

We have

$f_1 = x^2$, $f_1' = 2x$, $f_1'' = 2$

$f_2 = x^3$, $f_2' = 3x^2$, $f_2'' = 6x$

Plugging in we get:

$$x^2 f_1'' - 4x f_1' + 6 f_1$$

$$= x^2(2) - 4x(2x) + 6(x^2)$$

$$= 0$$

And,

$$x^2 f_2'' - 4x f_2' + 6 f_2$$

$$= x^2(6x) - 4x(3x^2) + 6(x^3)$$

$= 0$

Summary: Since f_1 and f_2 are linearly independent solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

on I, we know every solution on I is of the form

$$y_h = \underbrace{c_1 x^2 + c_2 x^3}_{c_1 f_1 + c_2 f_2}$$

where c_1, c_2 are any constants

Step 2: Now we need a particular solution y_p to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$.

Let's try

$$y_p = \frac{1}{12x} = \frac{1}{12} x^{-1}$$

Will find in topic 10

We plug it in.

$$y_p = \frac{1}{12} x^{-1}$$

$$y_p' = -\frac{1}{12} x^{-2}$$

$$y_p'' = \frac{2}{12} x^{-3} = \frac{1}{6} x^{-3}$$

We have:

$$x^2 y_p'' - 4x y_p' + 6 y_p$$

$$= x^2\left(\frac{1}{6}x^{-3}\right) - 4x\left(-\frac{1}{12}x^{-2}\right) + 6\left(\frac{1}{12}x^{-1}\right)$$

$$= \frac{1}{6}x^{-1} + \frac{1}{3}x^{-1} + \frac{1}{2}x^{-1}$$

$$= x^{-1} = \frac{1}{x}$$

It's a solution!

<u>Answer:</u> Every solution to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ is of the form

$$y = \underbrace{c_1 x^2 + c_2 x^3}_{\text{general solution } y_h \text{ to homogeneous}} + \underbrace{\frac{1}{12}x^{-1}}_{\text{particular solution } y_p \text{ to } x^2y''-4xy'+6y=\frac{1}{x}}$$

$$x^2y'' - 4xy' + 6y = 0$$