2150-01
2/24/25

(last part of topic 6...)

Previously we showed that the general solution to

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$ is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^{x}}_{y_p}$$

where c_1, c_2 are any constants.

So we get an infinite # of solutions to the differential equation, some solutions are:

$$y = \underbrace{0e^{2x} + 0e^{5x}}_{c_1 = 0,\ c_2 = 0} + 6e^{x} = 6e^{x}$$

$$y = \underbrace{e^{2x} - 12e^{5x}}_{c_1 = 1,\ c_2 = -12} + 6e^{x}$$

However, if you create an initial-value problem by specifying $y(x_0) = y_0$, $y'(x_0) = y_0'$ at some x_0, then there will only be one solution.

Ex: Solve

$$y'' - 7y' + 10y = 24e^{x}$$

$$y(0) = 0, \quad y'(0) = 1$$

$x_0 = 0$

The general solution to

$$y'' - 7y' + 10y = 24e^{x}$$

is

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

Let's make this also solve

$$y(0) = 0 \text{ and } y'(0) = 1$$

We have

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$

Need to solve:

$$\boxed{\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}} \longrightarrow \boxed{\begin{array}{l} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^{0} = 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^{0} = 1 \end{array}}$$

$e^0 = 1$

$$\boxed{\begin{aligned} c_1 + c_2 + 6 &= 0 \\ 2c_1 + 5c_2 + 6 &= 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= -5 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$.

Plug this into (2) to get:

$$2(-6 - c_2) + 5c_2 = -5$$

So, $-12 - 2c_2 + 5c_2 = -5$

Thus, $3c_2 = 7$

So, $\boxed{c_2 = 7/3}$

c_1

Then, $c_1 = -6 - c_2 = -6 - 7/3 = \boxed{-25/3}$

Answer: $\boxed{y = \frac{-25}{3} e^{2x} + \frac{7}{3} e^{5x} + 6e^{x}}$

$c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$

Topic 7 – 2nd order linear homogeneous constant coefficient

We will now learn methods to find the solutions to 2nd order equations. We start with the simplest ones we can. These are

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Where a_2, a_1, a_0 are constants, $a_2 \neq 0$

Ex:

$$y'' - 7y' + 10y = 0$$

$a_2 = 1$ $\quad a_1 = -7$ $\quad a_0 = 10$

Def: The **characteristic equation** of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

Ex: The characteristic equation of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

Why do we do this? The roots of the characteristic equation tell us the solution to the differential equation.

<u>Formula time</u> Consider

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (*)$$

where a_2, a_1, a_0 are constants and $a_2 \neq 0$. There are three cases depending on the roots of the characteristic equation $a_2 r^2 + a_1 r + a_0 = 0$.

<u>case 1:</u> If the characteristic equation has two distinct real roots r_1, r_2 then the solution to $(*)$ is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

<u>case 2:</u> If the characteristic equation has a repeated real root r, then the solution to $(*)$ is

$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

<u>case 3:</u> If the characteristic equation has imaginary roots $\alpha \pm i\beta$

α = alpha
β = beta
$i = \sqrt{-1}$

then the solution to $(*)$ is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Ex: Solve

$$y'' - 7y' + 10y = 0$$

characteristic equation:

$$r^2 - 7r + 10 = 0$$

The roots are:

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2} =$$

$$= 5, 2$$

case 1

two distinct real roots

$r_1 = 5,\ r_2 = 2$

Answer:

$$\boxed{y_h = c_1 e^{5x} + c_2 e^{2x}}$$

Ex: Solve

$$\boxed{y'' - 4y' + 4y = 0}$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are:

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2}$$

$$= \frac{4}{2} = 2$$

case 2
repeated
real
root
$r = 2$

Answer:

$$y_h = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{c_1 e^{rx} + c_2 x e^{rx}}$$