Math 2150-01
2/5/25

Topic 4 – Separable first order ODEs

<u>Def:</u> A first order ODE is called <u>separable</u> if it is of the form

$$\underbrace{N(y)}_{\text{just } y\text{'s and \#'s}} \cdot y' = \underbrace{M(x)}_{\text{just } x\text{'s and \#'s}}$$

or

$$N(y) \cdot \frac{dy}{dx} = M(x)$$

Ex:
$$\underbrace{y^2}_{N(y)} \cdot y' = \underbrace{2x^3}_{M(x)}$$

is separable

Ex: $\dfrac{dy}{dx} = \dfrac{x}{y}$

Multiply by y to get

$$\underbrace{y}_{N(y)} \cdot \frac{dy}{dx} = \underbrace{x}_{M(x)}$$

This is separable.

How to solve separable ODEs

Formal way	Informal way
$N(y) \cdot y' = M(x)$	$N(y) \cdot \frac{dy}{dx} = M(x)$
$\downarrow$	$\downarrow$
$N(y(x)) \cdot y'(x) = M(x)$	$N(y)\,dy = M(x)\,dx$
$\downarrow$	[differential form notation]
$\int N(y(x)) \cdot y'(x)\,dx = \int M(x)\,dx$	$\downarrow$
$u = y(x)$ $du = y'(x)dx$	$\boxed{\int N(y)\,dy = \int M(x)\,dx}$ Now integrate.
$\downarrow$	
$\int N(u)\,du = \int M(x)\,dx$ where $u = y$	

Ex: Find a solution to

$$y^2 \frac{dy}{dx} = x - 5$$

Also, on what interval I is the solution defined?

We have:

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2 \, dy = (x-5) \, dx$$

$$\int y^2 \, dy = \int (x-5) \, dx$$

$$\frac{1}{3} y^3 = \frac{1}{2} x^2 - 5x + C$$

$\times 3$

$$y^3 = \frac{3}{2}x^2 - 15x + 3C$$

$D = 3C$

$$y^3 = \frac{3}{2}x^2 - 15x + D$$

$$\boxed{y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}}$$

is a solution and it's defined on $I = (-\infty, \infty)$ that is any x is ok.

negative is ok for cube root

$$(-8)^{1/3} = -2$$

because $(-2)^3 = -8$

$\boxed{\text{Ex:}}$ Find a solution to

$$\frac{dy}{dx} + 2xy = 0$$

On what interval I does your solution exist?

This equation is linear so you could use topic 3 to solve it. But let's separate instead!

We have

$$\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = -2xy$$

$$\frac{1}{y}\,dy = -2x\,dx$$

$$\int \frac{1}{y}\,dy = \int (-2x)\,dx$$

$$\ln|y| = -x^2 + C$$

$$e^{\ln|y|} = e^{-x^2+C}$$

$$e^{\ln(z)} = z$$

$$|y| = e^{-x^2+C}$$

$$y = \pm e^{-x^2+C}$$

$$y = \pm e^{-x^2} e^{C}$$

$$y = \boxed{\pm e^{c}}\, e^{-x^2}$$

constant

$$y = De^{-x^2}$$

D is a constant

$$I = (-\infty, \infty)$$

TOPIC 3 METHOD

$$y' + 2xy = 0$$

$$A(x) = \int 2x\,dx = x^2$$

$$e^{x^2} y' + 2x e^{x^2} = 0$$

$$(e^{x^2} y)' = 0$$

$$e^{x^2} y = D \longrightarrow y = De^{-x^2}$$

$\boxed{\text{HW 4 } 1(c)}$ Find a solution to

$$\frac{dy}{dx} = \frac{-x}{y}$$

If possible, solve for y in your solution.
If you can do that find the interval I where the function is defined

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We have

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int (-x)\,dx$$

$\times 2$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

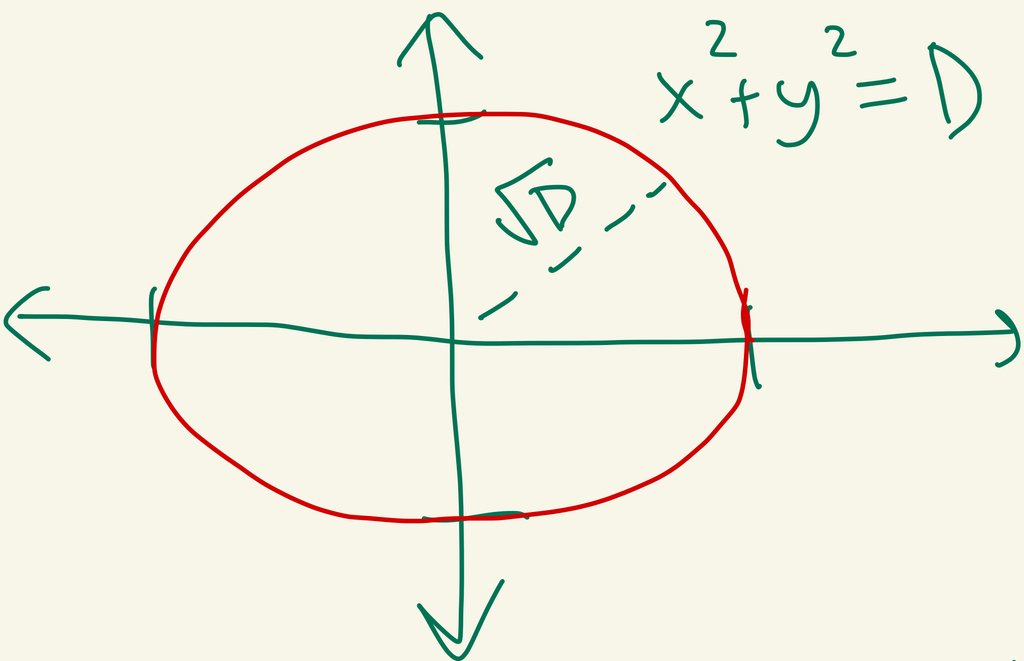
$$y^2 = -x^2 + 2C$$

$D = 2C$

$$y^2 = -x^2 + D$$

What if we don't solve for y?

This is $x^2 + y^2 = D$



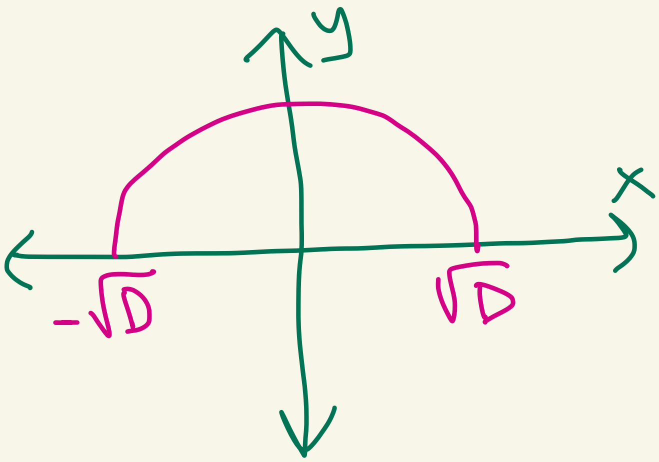
This is called an implicit solution to the ODE

Let's actually solve for y in $y^2 = -x^2 + D$.

We get
$$y = \pm\sqrt{-x^2 + D}$$

Solution 1

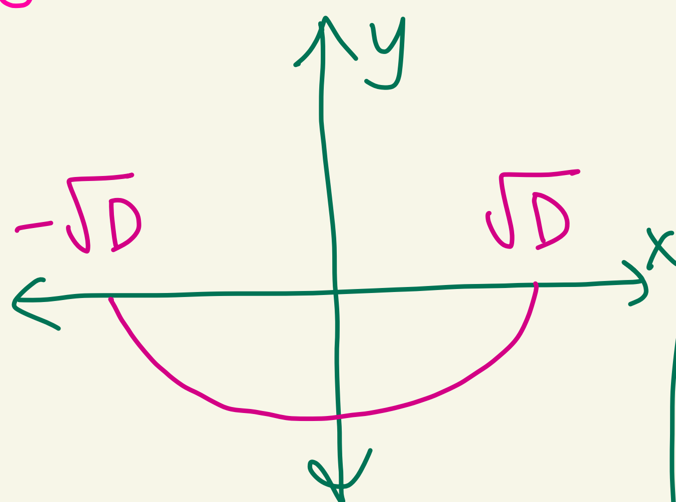
$$y = \sqrt{-x^2 + D}$$



$$I = [-\sqrt{D}, \sqrt{D}]$$

Solution 2

$$y = -\sqrt{-x^2 + D}$$



$$I = [-\sqrt{D}, \sqrt{D}]$$

$\boxed{\text{HW 4 } 1(d)}$ Find a solution to

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y(4) = 3$$

We know $y^2 = -x^2 + D$ solves $\frac{dy}{dx} = \frac{-x}{y}$. Let's make $\underbrace{y(4)=3}_{\text{when } x=4 \text{ we have } y=3}$.

Plug $x=4, y=3$ into $y^2 = -x^2 + D$.

We get: $3^2 = -(4)^2 + D$.

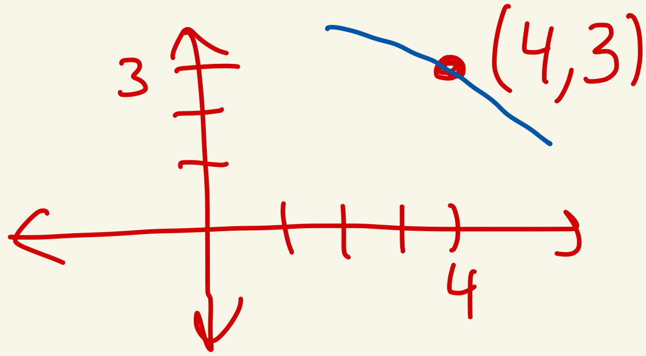
Then, $9 = -16 + D$

So, $D = 25$.

So, we get $y^2 = -x^2 + 25$

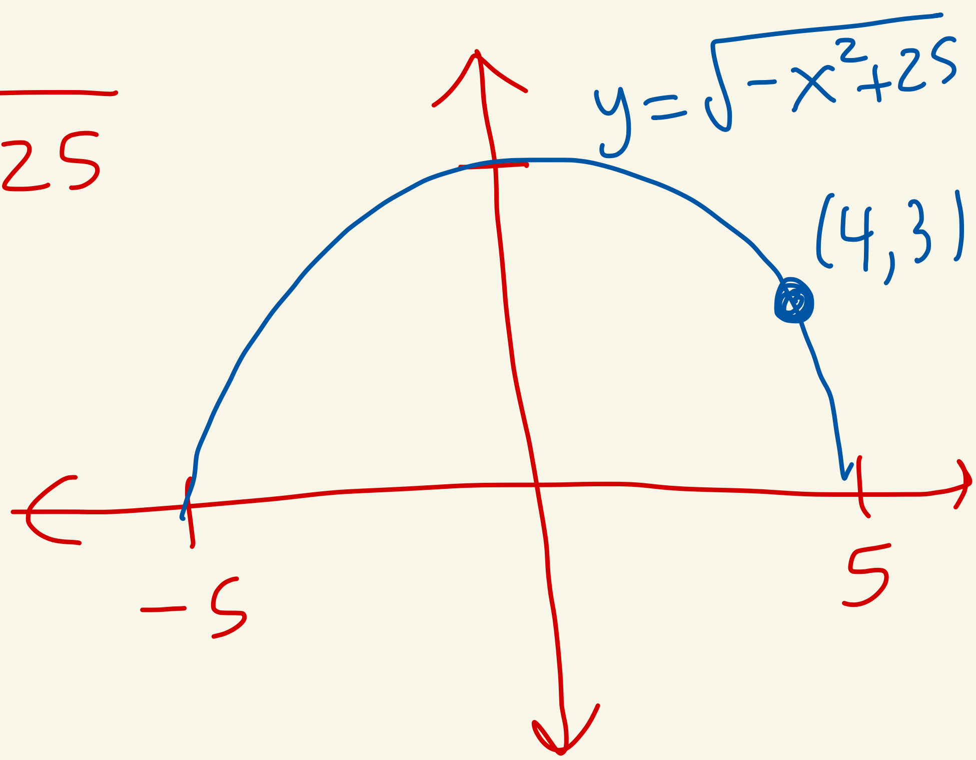
Then, $y = \pm\sqrt{-x^2+25}$

Need $y(4)=3$.



Pick + to make this happen.

$y = \sqrt{-x^2+25}$



$I = [-5, 5] \leftarrow \boxed{-5 \leq x \leq 5}$