Math 2150-01
3/3/25

Topic 8 – Method of undetermined coefficients

We want to be able to solve:

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

Where a_2, a_1, a_0 are constants.

Method:

① Find general solution y_h to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

topic 7

② Guess a particular solution y_p to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

topic 8

③ The general solution to

use topic

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

is

$$y = y_h + y_p$$

6

This table is on handout on the website. For guessing y_p.

Undetermined coefficients guess for y_p:

$b(x)$	y_p
constant	A
$5x-3$	$Ax+B$
$10x^2-x+1$	Ax^2+Bx+C
$\sin(6x)$	$A\cos(6x)+B\sin(6x)$
$\cos(6x)$	$A\cos(6x)+B\sin(6x)$
e^{3x}	Ae^{3x}
$(2x+1)e^{3x}$	$(Ax+B)e^{3x}$
x^2e^{3x}	$(Ax^2+Bx+C)e^{3x}$
$e^{3x}\sin(4x)$	$Ae^{3x}\cos(4x)+Be^{3x}\sin(4x)$
$e^{3x}\cos(4x)$	$Ae^{3x}\cos(4x)+Be^{3x}\sin(4x)$
$5x^2\sin(4x)$	$(Ax^2+Bx+C)\cos(4x)+(Dx^2+Ex+F)\sin(4x)$

Ex: Find the general solution to

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

Factor way:

$$(r+1)(r+2) = 0$$

$$r = -1, -2$$

The roots are:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2} = \boxed{-1, -2}$$

We get

$$\boxed{y_h = c_1 e^{-x} + c_2 e^{-2x}}$$

<u>Step 2:</u> Guess a solution y_p to

$$y'' + 3y' + 2y = 2x^2$$

Let's guess

$$y_p = Ax^2 + Bx + C$$

A, B, C unknown numbers

We have

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug these into the equation to get

$$\underbrace{(2A)}_{y_p''} + 3\underbrace{(2Ax+B)}_{y_p'} + 2\underbrace{(Ax^2+Bx+C)}_{y_p} = 2x^2$$

We get

$$2A+6Ax+3B+2Ax^2+2Bx+2C=2x^2$$

Regrouping:

$$\underbrace{\boxed{2A}}_{2}x^2+\underbrace{\boxed{(6A+2B)}}_{0}x+\underbrace{\boxed{(2A+3B+2C)}}_{0}=\underbrace{2x^2}_{2x^2+0x+0}$$

Get:

$$\boxed{\begin{aligned} 2A&=2 \quad ① \\ 6A+2B&=0 \quad ② \\ 2A+3B+2C&=0 \quad ③ \end{aligned}}$$

① gives $A=1$.

Plug into ② and get $6(1)+2B=0$.

$B=-3$

Plug into ③ and get $2(1)+3(-3)+2C=0$

$C=7/2$

Thus,

$$y_p=Ax^2+Bx+C=x^2-3x+\frac{7}{2}$$

Step 3: The general solution to

$$y'' + 3y' + 2y = 2x^2$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + \frac{7}{2}}$$

Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Solve

$$y'' - y' + y = 0$$

The characteristic equation is

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{1 \pm \sqrt{3}\, i}{2} = \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\alpha \pm \beta i}$$

$$\alpha = 1/2, \; \beta = \sqrt{3}/2$$

$$\boxed{\begin{array}{l} \underline{\text{General formula:}} \\ y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) \end{array}}$$

In our case we get:

$$y_h = c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

Step 2: Guess a solution to

$$y'' - y' + y = \underbrace{2\sin(3x)}_{b(x)}$$

Guess:

$$y_p = A\cos(3x) + B\sin(3x)$$

We get

$$y_p' = -3A\sin(3x) + 3B\cos(3x)$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x)$$

Plug these into the equation:

$$\left.\begin{array}{l} (-9A\cos(3x) - 9B\sin(3x)) \\ -(-3A\sin(3x) + 3B\cos(3x)) \\ +(A\cos(3x) + B\sin(3x)) \end{array}\right] \begin{array}{l} y_p'' \\ -y_p' \\ +y_p \end{array}$$

$$= 2\sin(3x)$$

Regrouping:

$$\underbrace{(3A - 8B)}_{2}\sin(3x) + \underbrace{(-8A - 3B)}_{0}\cos(3x)$$

$$= 2\sin(3x)$$

Need

$$\boxed{\begin{array}{l} 3A - 8B = 2 \\ -8A - 3B = 0 \end{array}} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

(2) gives $A = -\frac{3}{8}B$.

Plug into ①: $3\underbrace{\left(-\frac{3}{8}B\right)}_{A} - 8B = 2$

$$-\frac{9}{8}B - 8B = 2$$

$$-\frac{73}{8}B = 2$$

$$\boxed{B = \frac{-16}{73}}$$

So, $A = \frac{-3}{8}B = \frac{-3}{8}\left(\frac{-16}{73}\right) = \boxed{\frac{6}{73}}$

Thus,

$$y_p = \underbrace{\frac{6}{73}}_{A}\cos(3x) - \underbrace{\frac{16}{73}}_{B}\sin(3x)$$

<u>Step 3:</u> The general solution to

$$y'' - y' + y = 2\sin(3x)$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x)}$$