Math 2150-01
4/16/25

Ex: Let's find a power series solution to

$$\boxed{\begin{aligned} y' - 2xy &= 0 \\ y(0) &= 1 \end{aligned}}$$

center at $x_0 = 0$

coefficients

$$-2x = \overbrace{0 - 2x + 0x^2 + 0x^3 + \cdots}^{\text{power series centered at } x_0 = 0}$$

$$0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$$

$r = \infty$

This tells us that we will have a power series solution

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

will radius of convergence $r = \infty$

We need to find $y^{(n)}(0)$ for $n \geq 0$.

Given: $$\boxed{\begin{gathered} y' - 2xy = 0 \\ y(0) = 1 \end{gathered}} \leftarrow y' = 2xy$$

So, $\boxed{y(0) = 1}$

And, $y'(0) = 2[0][y(0)] = 2[0][1] = 0$

So, $\boxed{y'(0) = 0}$

Differentiate $y' = 2xy$ with respect to x to get:

$$y'' = 2y + 2xy'$$

$$\boxed{(fg)' = f'g + fg'}$$

So,

$$y''(0) = 2[y(0)] + 2(0)[y'(0)]$$
$$= 2[1] + 2(0)(0)$$
$$= 2.$$

So, $\boxed{y''(0) = 2}$

Differentiate $y'' = 2y + 2xy'$ to get

$$y''' = 2y' + 2y' + 2xy''$$
$$y''' = 4y' + 2xy''$$

So,

$$y'''(0) = 4[y'(0)] + 2(0)[y''(0)]$$
$$= 4(0) + 2(0)(2)$$
$$= 0$$

Thus, $\boxed{y'''(0) = 0}$

Differentiate $y''' = 4y' + 2xy''$ to get

$$y'''' = 4y'' + 2y'' + 2xy'''$$
$$= 6y'' + 2xy'''$$

So,

$$y''''(0) = 6\left[y''(0)\right] + 2(0)\left[y'''(0)\right]$$
$$= 6(2) + 2(0)(0)$$
$$= 12$$

Thus, $\boxed{y''''(0) = 12}$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$
$$+ \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \ldots$$

$$y(x) = 1 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 + \cdots$$

$$\boxed{y(x) = 1 + x^2 + \frac{1}{2}x^4 + \cdots}$$

with radius of convergence $r = \infty$

Side note Using topic 3, you can show

$$y(x) = e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots$$

$\boxed{t = x^2}$

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

Ex: Consider

$$\boxed{\begin{gathered} y'' + x^2 y' - (x-1)\,y = \ln(x) \\ y'(1) = 0, \quad y(1) = 0 \end{gathered}}$$

$x_0 = 1$

Coefficients

$$x^2 = 1 + 2(x-1) + (x-1)^2 + 0(x-1)^3 + \cdots$$

$$-(x-1) = 0 - 1\cdot(x-1) + 0(x-1)^2 + 0(x-1)^3 + \cdots$$

$r = \infty$

$$\ln(x) = -(x-1) + \tfrac{1}{2}(x-1)^2 + \cdots$$

$r = 1$

So we can find a solution

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots$$

with radius of convergence is at least $r = 1$.

We have

$$\boxed{\begin{aligned} y(1) &= 0 \\ y'(1) &= 0 \end{aligned}}$$

and

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

$$y''(1) = \ln(1) - (1)^2 [y'(1)] + (1-1)[y(1)]$$

$$= 0 - (1)[0] + (0)[0]$$

$$= 0$$

So,

$$y''(1) = 0$$

Differentiating above we get

$$y''' = \frac{1}{x} - 2xy' - x^2 y'' + (1)y + (x-1)y'$$

$$y'''(1) = \frac{1}{1} + 2(1)[y'(1)] - (1)^2[y''(1)] + y(1) + (1-1)[y'(0)]$$

$$= 1 + 2(1)[0] - (1)[0] + 0 + (0)(0)$$

$$= 1$$

Thus, $y'''(1) = 1$

One can calculate that

$$y''''(1) = -3$$

Thus,

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2$$

$$+ \frac{y'''(1)}{3!}(x-1)^3 + \frac{y'''(1)}{4!}(x-1)^4 + \cdots$$

$$= 0 + 0(x-1) + 0(x-1)^2$$

$$+ \frac{1}{3!}(x-1)^3 - \frac{3}{4!}(x-1)^4 + \cdots$$

So,

$$y(x) = \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots$$

with radius of convergence at least $r = 1$