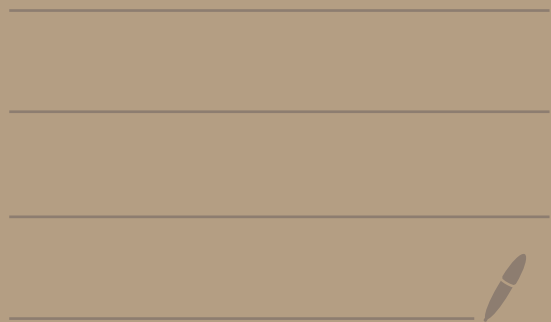


Math 2150-01

4/16/25



Ex: Let's find a power series solution to

$$y' - 2xy = 0$$

$$y(0) = 1$$

center at
 $x_0 = 0$



Coefficients

power series
centered at $x_0 = 0$

$$\begin{aligned} -2x &= 0 - 2x + 0x^2 + 0x^3 + \dots \\ 0 &= 0 + 0x + 0x^2 + 0x^3 + \dots \end{aligned}$$

$r = \infty$

This tells us that we will have a power series solution

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 \\ &\quad + \frac{y'''(0)}{3!}x^3 + \dots \end{aligned}$$

will radius of convergence $r = \infty$

We need to find $y^{(n)}(0)$ for $n \geq 0$.

Given:

$$\boxed{\begin{aligned} y' - 2xy &= 0 \\ y(0) &= 1 \end{aligned}} \leftarrow \boxed{y' = 2xy}$$

So, $\boxed{y(0) = 1}$

And, $y'(0) = 2[0][y(0)] = 2[0][1] = 0$

So, $\boxed{y'(0) = 0}$

Differentiate $y' = 2xy$ with respect to x to get:

$$y'' = 2y + 2xy'$$

$$\boxed{(fg)' = f'g + fg'}$$

So,

$$\begin{aligned}y''(0) &= 2[y(0)] + 2(0)[y'(0)] \\&= 2[1] + 2(0)(0) \\&= 2.\end{aligned}$$

So, $y''(0) = 2$

Differentiate $y'' = 2y + 2xy'$ to get

$$y''' = 2y' + 2y' + 2xy''$$

$$y''' = 4y' + 2xy''$$

So,

$$\begin{aligned}y'''(0) &= 4[y'(0)] + 2(0)[y''(0)] \\&= 4(0) + 2(0)(2) \\&= 0\end{aligned}$$

Thus, $y'''(0) = 0$

Differentiate $y''' = 4y' + 2xy''$ to get

$$\begin{aligned} y^{(4)} &= 4y'' + 2y'' + 2xy''' \\ &= 6y'' + 2xy''' \end{aligned}$$

So,

$$\begin{aligned} y^{(4)}(0) &= 6[y''(0)] + 2(0)[y'''(0)] \\ &= 6(2) + 2(0)(0) \\ &= 12 \end{aligned}$$

Thus, $y^{(4)}(0) = 12$

So,

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 \\ &\quad + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots \end{aligned}$$

$$y(x) = 1 + 0x + \frac{2}{2!} x^2 + \frac{0}{3!} x^3 + \frac{12}{4!} x^4 + \dots$$

$$y(x) = 1 + x^2 + \frac{1}{2} x^4 + \dots$$

With radius of convergence $r = \infty$

Side note Using topic 3,

you can show

$$y(x) = e^{x^2} = 1 + x^2 + \frac{1}{2} x^4 + \frac{1}{6} x^6 + \dots$$

$$\uparrow \boxed{t = x^2}$$

$$e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \dots$$

Ex: Consider

$$\boxed{\begin{aligned} y'' + x^2 y' - (x-1)y &= \ln(x) \\ y'(1) &= 0, y(1) = 0 \end{aligned}}$$

$\{x_0 = 1\}$

Coefficients

$$\begin{aligned} x^2 &= 1 + 2(x-1) + (x-1)^2 + 0(x-1)^3 + \dots \\ -(x-1) &= 0 - 1 \cdot (x-1) + 0(x-1)^2 + 0(x-1)^3 + \dots \\ \ln(x) &= -(x-1) + \frac{1}{2}(x-1)^2 + \dots \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} r = \infty \\ \\ r = 1 \end{array}$$

So we can find a solution

$$\begin{aligned} y(x) &= y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 \\ &\quad + \frac{y'''(1)}{3!}(x-1)^3 + \dots \end{aligned}$$

with radius of convergence
is at least $r=1$.

We have

$$y(1) = 0$$

$$y'(1) = 0$$

and

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

$$y''(1) = \ln(1) - (1)^2 [y'(1)] + (1-1)[y(1)]$$

$$= 0 - (1)[0] + (0)[0]$$

$$= 0$$

So,

$$y''(1) = 0$$

Differentiating above we get

$$y''' = \frac{1}{x} - 2xy' - x^2y'' + (1)y + (x-1)y'$$

$$y'''(1) = \frac{1}{1} + 2(1)[y'(1)] - (1)^2[y''(1)] \\ + y(1) + (1-1)[y'(0)]$$

$$= 1 + 2(1)[0] - (1)[0] \\ + 0 + (0)(0)$$

$$= 1$$

$$\text{Thus, } y'''(1) = 1$$

One can calculate that

$$y'''(1) = -3$$

Thus,

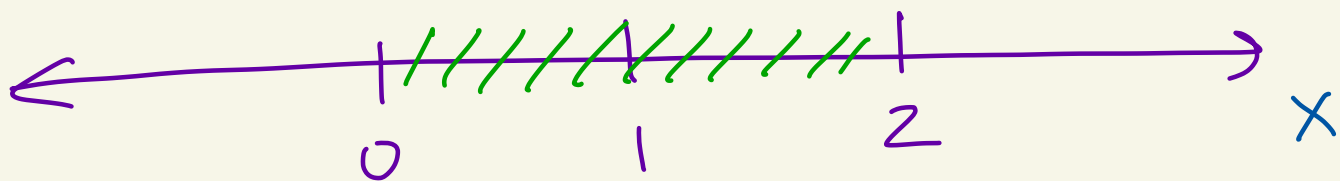
$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2$$

$$\begin{aligned}
 & + \frac{y'''(1)}{3!} (x-1)^3 + \frac{y^{(4)}(1)}{4!} (x-1)^4 + \dots \\
 & = 0 + 0(x-1) + 0(x-1)^2 \\
 & \quad + \frac{1}{3!} (x-1)^3 - \frac{3}{4!} (x-1)^4 + \dots
 \end{aligned}$$

So,

$$y(x) = \frac{1}{6} (x-1)^3 - \frac{1}{8} (x-1)^4 + \dots$$

with radius of convergence
at least $r=1$



converges at
least here