Math 2150-01
9/29/25

# Topic 8 - Method of Undetermined coefficients

We want to solve

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

Where $a_2, a_1, a_0$ are constants.

Ex: Two examples are:

$$y'' + 3y' + 2y = 2x^2$$

$$y'' - y' + y = 2\sin(3x)$$

## Method:

**Step 1:** Find the general solution $y_h$ to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

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**Step 2:** Find a particular solution $y_p$ to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

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**Step 3:** The general solution to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

is

$$y = y_h + y_p$$

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How do we find $y_p$ in step 2?
We guess $y_p$ and try it.
Here's a table for guessing.

| $b(x)$ | $y_p$ guess |
|---|---|
| constant number | $A$ |
| degree one polynomial like $2x-3$ or $5x$ | $Ax+B$ |
| degree two polynomial like $10x^2$ or $x^2-x+3$ | $Ax^2+Bx+C$ |
| $\sin(kx)$ where $k$ is constant | $A\cos(kx)+B\sin(kx)$ |
| $\cos(kx)$ where $k$ is constant | $A\cos(kx)+B\sin(kx)$ |

| | |
|---|---|
| exponential such as $e^{kx}$ or $3e^{kx}$ where $k$ is a constant | $Ae^{kx}$ |
| degree one poly. times exponential like $xe^{kx}$ or $(2x+3)e^{kx}$ where $k$ is a constant | $(Ax+B)e^{kx}$ |

Ex: Find the general solution to

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve the homogeneous equation:

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

$$\underbrace{(r+2)}_{\substack{r+2=0 \\ r=-2}}\underbrace{(r+1)}_{\substack{r+1=0 \\ r=-1}} = 0$$

The roots are $r = -1, -2$.

The general solution is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

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<u>Step 2:</u> Find a particular solution $y_p$ to

$$y'' + 3y' + 2y = \boxed{2x^2}$$

degree 2 polynomial

Guess: $y_p = Ax^2 + Bx + C$

We will plug it into the ODE and find $A, B, C$ that work.

We need:

$A, B, C$ are constants

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Now plug it into $y'' + 3y' + 2y = 2x^2$ to get:

$$\underbrace{(2A)}_{y_p''} + 3\underbrace{(2Ax + B)}_{y_p'} + 2\underbrace{(Ax^2 + Bx + C)}_{y_p} = 2x^2$$

We get:

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

$$\underbrace{\boxed{2A}}_{2}x^2 + \underbrace{\boxed{(6A + 2B)}}_{0}x + \underbrace{\boxed{(2A + 3B + 2C)}}_{0} = 2x^2$$

So we need:

$$\boxed{\begin{aligned} 2A &= 2 \\ 6A + 2B &= 0 \\ 2A + 3B + 2C &= 0 \end{aligned}}\quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

(1) gives $A = 1$.

Plug $A=1$ into (2) to get $6(1) + 2B = 0$

Thus, $B = -3$.

Plug $A=1, B=-3$ into (3) to get $2(1) + 3(-3) + 2C = 0$.

Thus, $C = 7/2$.

So,

$$y_p = Ax^2 + Bx + C$$

$$\boxed{y_p = x^2 - 3x + 7/2}$$

Step 3: The general solution to $y'' + 3y' + 2y = 2x^2$ is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + 7/2}$$

where $c_1, c_2$ can be any constants.

Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Solve the homogeneous equation $y'' - y' + y = 0$.

The characteristic equation is

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{1 \pm \sqrt{3}\, i}{2}$$

$i = \sqrt{-1}$

$$= \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\alpha \pm \beta i}$$

$\alpha \pm \beta i$

$\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$

roots

$\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$$\boxed{y_h = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)}$$

where $c_1, c_2$ are any constants