Math 2150-02
2/10/25

I put a study guide with practice tests on the website for test 1 which is on March 17

Topic 5 – First order exact equations

Suppose you have a first order equation of the form:

$$\underbrace{M(x,y)} + \underbrace{N(x,y)} \cdot y' = 0$$

expressions with x's & y's

Ex:

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

Further suppose there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

$\boxed{\text{Ex:}}$

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}\, y' = 0$$

$$f(x,y) = x^2 y - y$$

$$\frac{\partial f}{\partial x} = 2xy + 0 = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

Suppose $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$.

Then,

$$\underbrace{M(x,y)} + \underbrace{N(x,y)} \cdot \underbrace{y'} = 0$$

becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

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f(x,y) is a function of x, y

y = y(x) is a function of x

chain rule:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} \cdot \frac{d}{dx}(y)$$

$$= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

So, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

becomes $\frac{df}{dx} = 0$.

So for example the family of curves $f(x,y) = c$ where c is a constant will satisfy $\frac{df}{dx} = 0$.

<u>Summary:</u> If $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$, then the family of curves $f(x,y) = c$ where c is any constant will give an implicit solution to

$$M(x,y) + N(x,y) \cdot y' = 0$$

When such an f exists we call the equation

$$M(x,y) + N(x,y) \cdot y' = 0$$

an <u>exact</u> equation

<u>Ex:</u> Consider

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)} y' = 0$$

Let

$$f(x,y) = x^2 y - y$$

Then

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

So, $2xy+(x^2-1)y'=0$ is exact and a family of implicit solutions is given by

$$\boxed{x^2y-y=c} \leftarrow f(x,y)=c$$

where c is any constant.

Check #1

Suppose $x^2y-y=c$.
Differentiate both sides with respect to x to get:

$$2xy+x^2\underbrace{y'}_{\frac{dy}{dx}}-\underbrace{y'}_{\frac{dy}{dx}}=0$$

$$2xy+(x^2-1)y'=0$$

original equation

$\boxed{\text{Check \#2}}$ You can solve for y in our solution $x^2y - y = c$.

You get: $y = \dfrac{c}{x^2 - 1}$

Does this satisfy our equation?

$$y = c(x^2-1)^{-1}$$

$$y' = -c(x^2-1)^{-2}\cdot(2x)$$

$$= \frac{-2cx}{(x^2-1)^2}$$

Plug these in to get:

$$2xy + (x^2-1)y'$$

$$= 2x\underbrace{\left(\frac{c}{x^2-1}\right)}_{y} + (x^2-1)\underbrace{\left(\frac{-2cx}{(x^2-1)^2}\right)}_{y'}$$

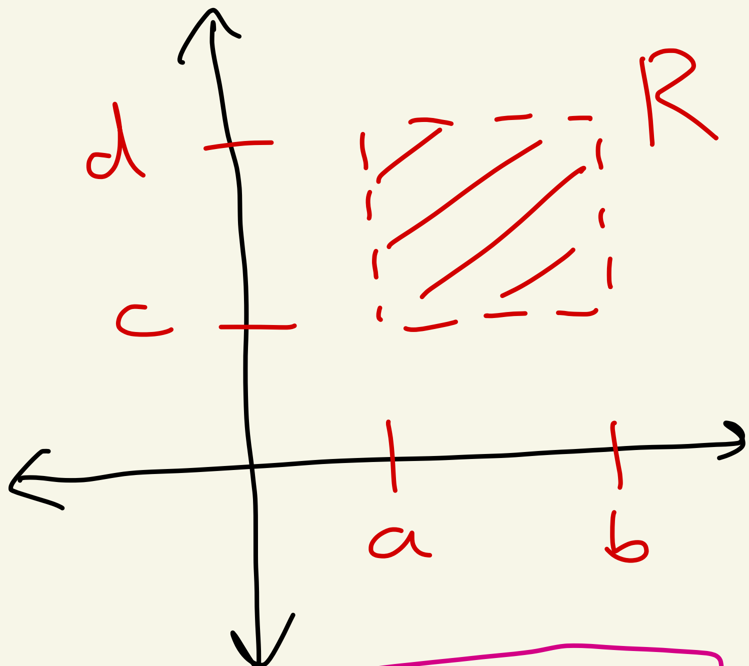
$$= \frac{2cx}{x^2-1} + \frac{-2cx}{x^2-1} = 0$$

So, $y = \dfrac{c}{x^2-1}$ satisfies

$$2xy + (x^2-1)y' = 0$$

When does such an f exist?

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<y<d$.



Then,

$M(x,y)+N(x,y)y'=0$

will be exact if and only if

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Here a,b,c,d can be $\pm\infty$

Proof: See onlines notes if interested

Ex: Consider the previous equation

$$\underbrace{2xy}_{M}+\underbrace{(x^2-1)}_{N}y'=0$$

$$M(x,y)=2xy$$
$$N(x,y)=x^2-1$$

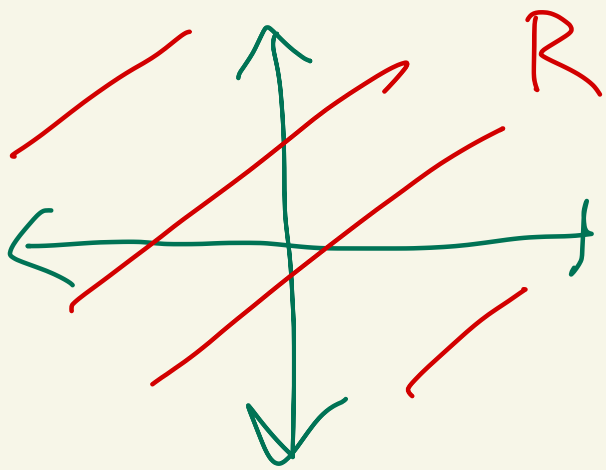
these are continuous everywhere

$$\frac{\partial M}{\partial x}=2y \qquad \frac{\partial N}{\partial x}=2x$$

$$\frac{\partial M}{\partial y}=2x \qquad \frac{\partial N}{\partial y}=0$$

continuous everywhere

R is the entire xy-plane



And, $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$!!!

Thus, $2xy + (x^2-1)y' = 0$

is exact. That is, there will exist f with

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$