Math 2150-02
2/12/25

Last time we saw that

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}y' = 0$$

is exact.

We had $f(x,y) = x^2y - y$

and $x^2y - y = c$ solved the equation.

But how would you find such an f?

Suppose we don't know what f is.

The f needs to satisfy

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M(x,y) \\ \frac{\partial f}{\partial y} &= N(x,y) \end{aligned}} \longrightarrow \boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2-1 \quad (2) \end{aligned}}$$

use these equations to find f

Pick 1 to start with:

$$\frac{\partial f}{\partial x} = 2xy$$

Integrate with respect to x to get:

$$f(x,y) = x^2 y + \underbrace{C(y)}_{\text{constant with respect to } x\text{, that is it only has \#'s and } y\text{'s}}$$

We want to plug this into equation 2 so first differentiate with respect to y to get:

$$\frac{\partial f}{\partial y} = x^2 + C'(y)$$

Set this equal to equation 2 which said that $\frac{\partial f}{\partial y} = x^2 - 1$.

We get:

$$x^2 + C'(y) = x^2 - 1$$

So,

$$C'(y) = -1$$

Thus,

$$C(y) = -y + D$$

D is a number

Ergo,

$$f(x,y) = x^2 y + C(y)$$

$$= x^2 y - y + D$$

You can make $D = 0$ because your just gonna set $f(x,y) = \text{constant}$

If you did

$$x^2 y - y + D = \text{constant}$$

$$x^2 y - y = \underbrace{\text{constant} - D}_{\text{constant}}$$

Answer:

$$\boxed{f(x,y) = x^2 y - y}$$

HW 5 | 2(b)

Consider the initial value problem

$$(e^x + y) + (2 + x + ye^y)y' = 0$$

$$y(0) = 1$$

First let's solve

$$\underbrace{(e^x + y)}_{M} + \underbrace{(2 + x + ye^y)}_{N} y' = 0$$

Let's show the equation is exact.

$$M(x,y) = e^x + y$$
$$N(x,y) = 2 + x + ye^y$$

these are continuous everywhere for all x, y

$\frac{\partial M}{\partial x} = e^x$	$\frac{\partial N}{\partial x} = 1$
$\frac{\partial M}{\partial y} = 1$	$\frac{\partial N}{\partial y} = e^y + ye^y$

<u>Check:</u> $\frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}$

So the equation is exact.
Thus, there will be an $f(x,y)$ that satisfies:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N \end{aligned}}$$

Start with ①:

$$\frac{\partial f}{\partial x} = e^x + y$$

Integrate with respect to x to get:

$$f(x,y) = e^x + yx + \underbrace{C(y)}_{\text{constant with respect to } x}$$

We want to plug this into ② so we first take the y derivative and we get:

$$\frac{\partial f}{\partial y} = x + C'(y)$$

Set this equal to ② which says $\frac{\partial f}{\partial y} = 2 + x + ye^y$ and we get

$$x + C'(y) = 2 + x + ye^y$$

So,

$$C'(y) = 2 + ye^y$$

This results in

$$C(y) = \int (2 + ye^y)\,dy$$

$$= 2y + \underbrace{\int ye^y\,dy}$$

LIATE

$\int u\,dv = uv - \int v\,du$

$u = y \qquad du = dy$

$dv = e^y\,dy \qquad v = e^y$

$$= 2y + \left(ye^y - \int e^y\,dy \right)$$

$$= 2y + ye^y - e^y$$

You don't need to add a constant since we will set our f = constant

Thus,

$$f(x,y) = e^x + yx + C(y)$$
$$= e^x + yx + \underbrace{2y + ye^y - e^y}_{C(y)}$$

So, a solution to

$$(e^x + y) + (2 + x + ye^y)y' = 0$$

is given by

$$\underbrace{\boxed{e^x + yx + 2y + ye^y - e^y = c}}_{f(x,y)=c}$$ ← ODE solution

where c is a constant.

This is called an implicit solution to the ODE because we can't solve for y, we just have an equation relating y and x.

Now let's make $y(0)=1$.

Plug in $x=0$ and $y=1$ into our solution to find c.

We get:

$$\underbrace{e^0}_{1} + \underbrace{(1)(0)}_{0} + \underbrace{2(1)}_{2} + \underbrace{(1)e^1 - e^1}_{0} = c$$

So, $c=3$.

$$\boxed{\text{Answer to the initial value problem is: } e^x + yx + 2y + ye^y - e^y = 3}$$