Math 2150-02

2/19/25

(Topic 6 continued...)

For the remainder of this topic we will learn the theory of solving linear 2nd order

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)}$$

on some interval I where $a_2(x), a_1(x), a_0(x), b(x)$ are all continuous on I and $a_2(x) \neq 0$ on I.

We will assume these conditions for the rest of the section.

$\boxed{\text{Fact 1:}}$ If $f_1(x)$ and $f_2(x)$ are two linearly independent solutions to the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad (*)}$$

on I, then every solution to $(*)$ on I is of the form

$$\boxed{y_h = c_1 f_1(x) + c_2 f_2(x)}$$

where c_1, c_2 are constants.

$\boxed{\text{Fact 2:}}$ Suppose we can find a particular solution y_p to

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x) \quad (**)}$$

on I. Then every solution to $(**)$ on I is of the form

$$y = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{y_h} + y_p$$

(solution to homogeneous equation)

Ex:

Let's find all solutions to

$$y'' - 7y' + 10y = 0$$

on $I = (-\infty, \infty)$.

Consider

$$f_1(x) = e^{2x}$$

and

next section we will see how to

$f_2(x) = e^{5x}$ find these

Last time we showed these functions are linearly independent on I.

Claim 1: f_1 solves $y'' - 7y' + 10y = 0$

We have $f_1(x) = e^{2x}$, $f_1'(x) = 2e^{2x}$, $f_1''(x) = 4e^{2x}$.

Plugging in we get:

$$f_1'' - 7f_1' + 10f_1$$

$$= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x})$$

$$= (4 - 14 + 10)e^{2x} = 0$$

Claim 2: f_2 solves $y'' - 7y' + 10y = 0$

We have $f_2(x) = e^{5x}$, $f_2'(x) = 5e^{5x}$,

$f_2''(x) = 25e^{5x}$.

Plugging into the equation we get:

$$f_2'' - 7f_2' + 10f_2$$
$$= 25e^{5x} - 7(5e^{5x}) + 10(e^{5x})$$
$$= (25 - 35 + 10)e^{5x}$$
$$= 0$$

Thus, $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are two linearly independent solutions to

$$y'' - 7y' + 10y = 0$$

$= 0$ homogeneous

So every solution to

$$y'' - 7y' + 10y = 0$$

is of the form

$$y_h = c_1 f_1(x) + c_2 f_2(x)$$

homogeneous

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

where c_1, c_2 are any constants.

What about

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$?

Consider

$$\underbrace{y_p}_{\text{particular solution}} = 6e^x$$

later we will learn how to find this

Let's show that y_p solves

$$y'' - 7y' + 10y = 24e^x$$

We have

$$y_p = 6e^x, \quad y_p' = 6e^x, \quad y_p'' = 6e^x$$

Plug it in to the left-side:

$$y_p'' - 7y_p' + 10y_p$$

$$= 6e^x - 7(6e^x) + 10(6e^x)$$

$$= 24e^x$$

<u>Summary:</u> Every solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$ is of the form

$$y = y_h + y_p$$

$$= c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

general solution y_h to homogeneous $y''-7y'+10y=0$

particular solution y_p to $y''-7y'+10y=24e^x$

Ex: Let's find all the solutions to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$

Step 1: We need to solve the homogoneous equation

$$x^2 y'' - 4xy' + 6y = 0 \qquad (*)$$

Consider $f_1(x) = x^2$, $f_2(x) = x^3$.

Let's show f_1 and f_2 are linearly independent solutions to $(*)$.

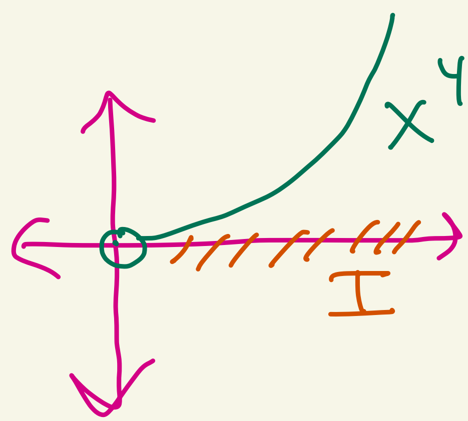
Let's check independence:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

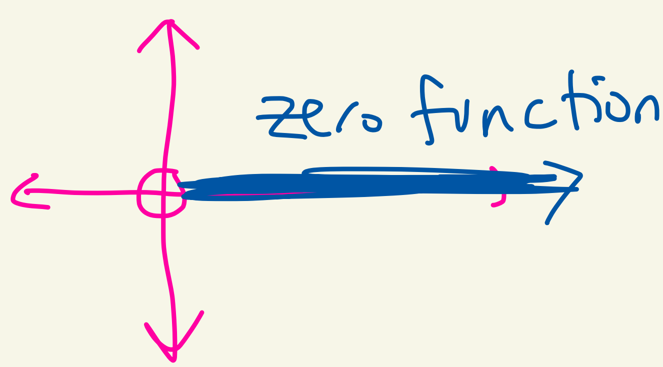
$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= (x^2)(3x^2) - (x^3)(2x)$$

$$= x^4$$



This is not the zero function on $I = (0, \infty)$.



So, f_1 and f_2 are linearly independent on $I = (0, \infty)$.

Now we must verify that f_1 and f_2 solve the homogeneous equation.

We have:

$f_1 = x^2,\ f_1' = 2x,\ f_1'' = 2$

$f_2 = x^3,\ f_2' = 3x^2,\ f_2'' = 6x$

Plug them individually into

$$x^2 y'' - 4xy' + 6y = 0$$

We get

$$x^2 f_1'' - 4x f_1' + 6 f_1$$

$$= x^2(2) - 4x(2x) + 6(x^2)$$

$$= 0$$

And

$$x^2 f_2'' - 4x f_2' + 6 f_2$$

$$= x^2(6x) - 4x(3x^2) + 6(x^3)$$

$$= 0$$

Thus, f_1 and f_2 are linearly independent solutions to the homogeneous equation

$$x^2 y'' - 4xy' + 6y = 0$$

Thus every solution is of the form

$$y_h = \underbrace{c_1 x^2 + c_2 x^3}_{c_1 f_1 + c_2 f_2}$$

where c_1, c_2 are constants.

<u>Step 2:</u> We need a particular solution y_p to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I=(0,\infty)$

Let's try $y_p = \frac{1}{12}x^{-1}$] Topic 10 to find this

Let's plug it in.

We have:

$$y_p = \frac{1}{12}x^{-1}, \quad y_p' = \frac{-1}{12}x^{-2}, \quad y_p'' = \frac{2}{12}x^{-3}$$

Plug into left-side:

$$x^2 y_p'' - 4x y_p' + 6y_p$$

$$= x^2\left(\frac{1}{6}x^{-3}\right) - 4x\left(\frac{-1}{12}x^{-2}\right) + 6\left(\frac{1}{12}x^{-1}\right)$$

$$= \frac{1}{6}x^{-1} + \frac{1}{3}x^{-1} + \frac{1}{2}x^{-1}$$

$$= x^{-1} = \frac{1}{x}$$

So, $y_p = \frac{1}{12}x^{-1}$ solves

$$x^2y'' - 4xy' + 6y = \frac{1}{x}$$

Thus, every solution to this equation is of the form

$$y = y_h + y_p$$

$$= \underbrace{c_1 x^2 + c_2 x^3}_{\substack{\text{general solution} \\ y_h \text{ to homogeneous} \\ x^2y''-4xy'+6y=0}} + \underbrace{\frac{1}{12}x^{-1}}_{\substack{\text{particular} \\ \text{solution } y_p \text{ to} \\ x^2y''-4xy'+6y=\frac{1}{x}}}$$