2150-02
2/24/25

(Last part of topic 6...)

Last time we showed that the general solution to

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$ is

$$y = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^x}_{y_p}$$

where c_1, c_2 are any constants.

This gives an infinite number of solutions. For example, some are:

$$y = 3e^{2x} - 2e^{5x} + 6e^x$$

$c_1 = 3$
$c_2 = -2$

$$y = 0e^{2x} + 1 \cdot e^{5x} + 6e^{x}$$
$$= e^{5x} + 6e^{x}$$

$c_1 = 0$
$c_2 = 1$

$$y = 6e^{x}$$

$c_1 = 0, \; c_2 = 0$

If you constrain the problem to have initial conditions then you will get only one solution.

Ex: Solve

$$y'' - 7y' + 10y = 24e^{x}$$
$$y(0) = 0, \; y'(0) = 1$$

We know any solution to

$$y'' - 7y' + 10y = 24e^x$$

must be of the form

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

Let's make this function satisfy $y(0) = 0$, $y'(0) = 1$.

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$$

$$y' = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x$$

$$\boxed{\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}} \longrightarrow \boxed{\begin{array}{l} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^0 = 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^0 = 1 \end{array}}$$

$e^0 = 1$

$$\boxed{\begin{array}{l} c_1 + c_2 + 6 = 0 \\ 2c_1 + 5c_2 + 6 = 1 \end{array}}$$

$$\boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= -5 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$.

Plug this into (2) to get:

$$2(-6 - c_2) + 5c_2 = -5$$

So, $-12 - 2c_2 + 5c_2 = -5$

So, $3c_2 = 7$

So, $c_2 = 7/3$

Thus, $c_1 = -6 - c_2 = -6 - 7/3 = -25/3$

Answer:

$$\boxed{y = \frac{-25}{3} e^{2x} + \frac{7}{3} e^{5x} + 6e^{x}}$$

$\boxed{\text{In summary:}}$ Let I be an interval with a_2, a_1, a_0, b continuous on I and $a_2(x) \neq 0$ on I, then

$$\boxed{\begin{array}{c} a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x) \\ y(x_0) = y_0, \quad y'(x_0) = y_0' \end{array}}$$

(where x_0 is in I)

has only one solution.

Starting in topic 7 we learn the techniques to find the solutions to 2nd order equations.

$\boxed{\text{Topic 7 – 2nd order linear homogeneous constant coefficient}}$

We will learn how to solve equations of the form

$$a_2 y'' + a_1 y' + a_0 y = 0$$

where a_2, a_1, a_0 are numbers where $a_2 \neq 0$

Ex: $y'' - 7y' + 10y = 0$

$a_2 = 1$ $a_1 = -7$ $a_0 = 10$

Def: The characteristic equation of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

Ex: The characteristic eqn. of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

It turns out that the roots of the characteristic equation give you the solution to the differential equation

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$\boxed{\text{Formula}}$ Consider

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (*)$$

where a_2, a_1, a_0 are constants and $a_2 \neq 0$. There are three cases depending on the roots of the characteristic equation $a_2 r^2 + a_1 r + a_0 = 0$.

<u>Case 1</u>: If the characteristic equation has two distinct real roots r_1, r_2, then the solution to $(*)$ is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: If the characteristic equation has a repeated real root r, then the solution to (*) is

$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

Case 3: If the characteristic equation has two complex roots $\alpha \pm \beta i$, then the solution to (*) is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$\alpha \leftarrow$ alpha

$\beta \leftarrow$ beta

$i = \sqrt{-1}$

$i^2 = -1$

Ex: Consider

$$y'' - 7y' + 10y = 0$$

Characteristic equation:

$$r^2 - 7r + 10 = 0$$

The roots are:

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2}$$

$$= \frac{10}{2}, \frac{4}{2}$$

$$= 5, 2$$

two distinct real roots $r_1 = 5, r_2 = 2$

Answer:

$$y_h = \underbrace{c_1 e^{5x} + c_2 e^{2x}}_{c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

Ex: Consider

$$y'' - 4y' + 4y = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = 2$$

one real root
$r = 2$

<u>Answer:</u>

$$y_h = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{c_1 e^{rx} + c_2 x e^{rx}}$$