2150-02
2/26/25

Ex: Solve

$$y'' - 4y' + 13y = 0$$

The characteristic polynomial is

$$r^2 - 4r + 13 = 0$$

The roots are:

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\,\overbrace{\sqrt{-1}}^{i}}{2}$$

$$= \frac{4 \pm 6i}{2} = \underbrace{2 \pm 3i}_{\text{two imaginary roots}}$$

$$= 2+3i,\ 2-3i$$

Last time formula:

$\alpha \pm \beta i \leftarrow$ roots

Solution:

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

In our problem

$$\alpha \pm \beta i = 2 \pm 3i$$

So, $\alpha = 2$, $\beta = 3$.

Summary: The general solution to

$$y'' - 4y' + 13y = 0$$

is

$$y_h = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

HW 7
2(a)

Solve

$$\boxed{\begin{gathered}4y'' - y' = 0 \\ y'(0) = 0 \\ y(0) = 0\end{gathered}}$$

First we solve:

$$4y'' - y' = 0$$

The characteristic equation is

$$4r^2 - r = 0$$

Method 1

$$r(4r-1) = 0$$

Method 2

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(0)}}{2(4)}$$

$r = 0, \frac{1}{4}$

$$= \frac{1 \pm \sqrt{1}}{8} = \frac{1 \pm 1}{8}$$

$$= \frac{1+1}{8}, \frac{1-1}{8}$$

$$= \frac{1}{4}, 0$$

We get two distinct real roots: $1/4, 0$.

Thus, the general solution to

$$4y'' - y' = 0$$

is

$$y_h = c_1 e^{\frac{1}{4}x} + c_2 e^{0x}$$

$e^0 = 1$

$$= c_1 e^{x/4} + c_2$$

Now we want our solution

to also satisfy $y(0)=0, y'(0)=0$

We have

$$y_h = c_1 e^{x/4} + c_2$$

$$y_h' = \frac{1}{4} c_1 e^{x/4}$$

Need to solve:

$$\boxed{\begin{aligned} y_h'(0) &= 0 \\ y_h(0) &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} \tfrac{1}{4} c_1 e^{0/4} &= 0 \\ c_1 e^{0/4} + c_2 &= 0 \end{aligned}}$$

$$e^{0/4} = e^0 = 1$$

$$\boxed{\begin{aligned} \tfrac{1}{4} c_1 &= 0 \quad (1) \\ c_1 + c_2 &= 0 \quad (2) \end{aligned}}$$

(1) tells us that $c_1 = 0$

(2) gives $c_2 = -c_1 = -(0) = 0$

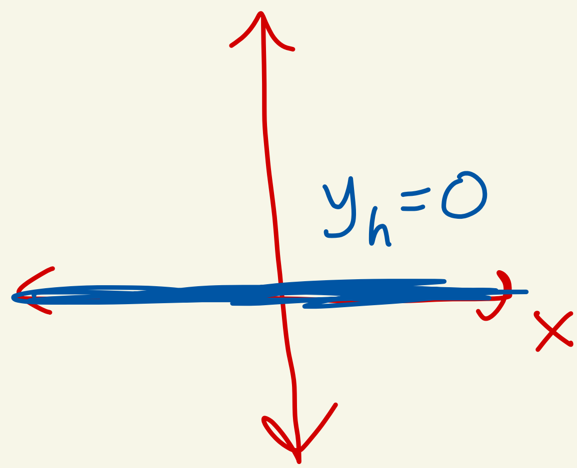
$\boxed{(2)\ c_1 + c_2 = 0}$

So need $c_1 = 0,\ c_2 = 0$.

Answer:

$$y_h = c_1 e^{x/4} + c_2$$
$$= 0e^{x/4} + 0$$
$$= 0$$

$\boxed{y_h = 0 \text{ is answer}}$



Let me show you why case 1 works.

Consider $a_2 y'' + a_1 y' + a_0 y = 0$.

Let's try guessing $y = e^{rx}$ where r is a real number.

We have

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

Plug this into the LHS of the differential equation:

$$\begin{aligned} & a_2 y'' + a_1 y' + a_0 y \\ &= a_2(r^2 e^{rx}) + a_1(re^{rx}) + a_0 e^{rx} \\ &= e^{rx}\left[a_2 r^2 + a_1 r + a_0\right] \end{aligned}$$

the characteristic equation

The above will be zero when r is a root of the characteristic equation, that is when $a_2 r^2 + a_1 r + a_0 = 0$

So, in case 1 of our theorem, if r_1 and r_2 are distinct real roots of $a_2 r^2 + a_1 r + a_0 = 0$ then $e^{r_1 x}$ and $e^{r_2 x}$ will be solutions to $a_2 y'' + a_1 y' + a_0 y = 0$.

Let's show these solutions are linearly independent.
We have

$$W(e^{r_1 x}, e^{r_2 x}) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= \left(e^{r_1 x}\right)\left(r_2 e^{r_2 x}\right) - \left(e^{r_2 x}\right)\left(r_1 e^{r_1 x}\right)$$

$$= r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$$

$$= \underbrace{(r_2 - r_1)}_{\substack{\text{not} \\ \text{zero} \\ \text{since} \\ r_1 \neq r_2}} \underbrace{e^{(r_1+r_2)x}}_{\substack{\text{never} \\ \text{zero}}} \neq 0$$

for all x.

Thus, $e^{r_1 x}$ and $e^{r_2 x}$ are two linearly independent solutions to $a_2 y'' + a_1 y' + a_0 y = 0$, so by topic 6, all solutions are of the form

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
