Math 2150-02
2/3/25

There was a mistake in HW 1 problem 2(a) solutions in the f_2 calculation. I fixed it and re-uploaded the solutions.

$\boxed{\text{Continue topic 3}}$

$\boxed{\text{HW 1(d)}}$

Solve

$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$

on $I = (-\infty, \infty)$

We have

$$y' + \underbrace{2x}_{\text{integrate this}}y = xe^{-x^2}$$

$$A(x) = \int 2x\,dx = x^2$$

multiply the ODE by $e^{A(x)} = e^{x^2}$

to get

$$\underline{\underline{e^{x^2}}}y' + \underline{\underline{e^{x^2}}}(2xy) = \underline{\underline{e^{x^2}}} x e^{-x^2}$$

$$\underbrace{e^{x^2}y' + e^{x^2}(2xy)}_{\text{left-side always becomes } (e^{A(x)}\cdot y)'}$$

$$(e^{x^2}y)' = x\underbrace{e^{x^2-x^2}}_{e^0=1}$$

So we get

$$(e^{x^2}\cdot y)' = x$$

Integrate both sides with respect to x (to get at the y) to get:

$$e^{x^2}\cdot y = \int x\,dx$$

$$e^{x^2} \cdot y = \frac{x^2}{2} + C$$

Divide by e^{x^2} (or multiply by e^{-x^2}) to get:

$$y = e^{-x^2}\left(\frac{x^2}{2} + C\right)$$

$$\boxed{y = \frac{1}{2}x^2 e^{-x^2} + Ce^{-x^2}}$$

All solutions to $y' + 2xy = xe^{-x^2}$ are of the form

$$y = \frac{1}{2}x^2 e^{-x^2} + Ce^{-x^2}$$

Ex: Let's solve

$$y' + \underbrace{\cos(x)}_{\text{integrate this}} y = \sin(x)\cos(x)$$

on $I = (-\infty, \infty)$

Let

$$A(x) = \int \cos(x)\,dx = \sin(x)$$

Multiply the ODE by $e^{A(x)} = e^{\sin(x)}$ to get:

$$\underbrace{e^{\sin(x)} y' + e^{\sin(x)} \cos(x) y}_{\text{always equals } (e^{A(x)} y)'} = e^{\sin(x)} \sin(x)\cos(x)$$

We get

$$\left(e^{\sin(x)} \cdot y\right)' = e^{\sin(x)} \sin(x)\cos(x)$$

Integrate both sides with respect to x to get:

$$e^{\sin(x)} \cdot y = \int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$\int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$= \int e^t \cdot t\,dt = \int t e^t\,dt$$

$$\boxed{\begin{aligned} t &= \sin(x) \\ dt &= \cos(x)\,dx \end{aligned}}$$

$$= t e^t - \int e^t\,dt$$

$$\boxed{\text{LIATE}}$$

$$\boxed{\begin{aligned} u &= t & du &= dt \\ dv &= e^t\,dt & v &= e^t \end{aligned}}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= te^{t} - e^{t} + C$$

$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

So we get

$$e^{\sin(x)} \cdot y = \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

Multiply both sides by $e^{-\sin(x)}$ to get

$$\cancel{e^{-\sin(x)}e^{\sin(x)}}y = \sin(x)\cancel{e^{-\sin(x)}e^{\sin(x)}} - \cancel{e^{-\sin(x)}e^{\sin(x)}} + Ce^{-\sin(x)}$$

$$y = \sin(x) - 1 + Ce^{-\sin(x)}$$

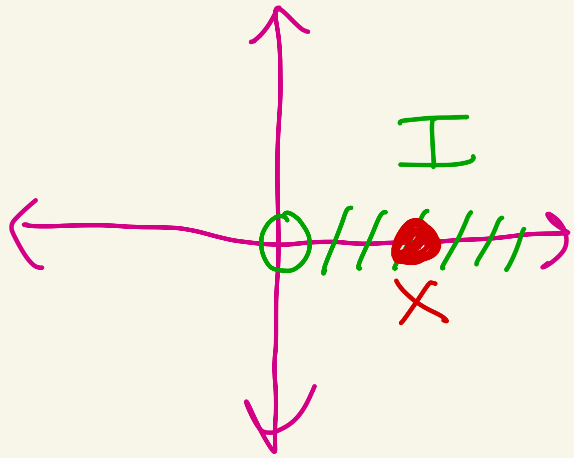
Answer:

$$\boxed{y = \sin(x) - 1 + Ce^{-\sin(x)}}$$

<u>Ex:</u> Solve

$$xy' + y = 3x^3 + 1$$

on $I = (0, \infty)$



We have

$$\underbrace{x}_{\text{need a 1 here}} y' + y = 3x^3 + 1$$

Divide by x to get:

$$y' + \underbrace{\frac{1}{x}}_{\text{integrate this}} y = 3x^2 + \frac{1}{x}$$

$$A(x) = \int \frac{1}{x}\,dx = \ln|x|$$
$$= \ln(x)$$

assuming $x > 0$ since x is in $I = (0, \infty)$

And

$$e^{A(x)} = e^{\ln(x)} = x$$

$$e^{\ln(z)} = z$$

Multiply $y' + \frac{1}{x}y = 3x^2 + \frac{1}{x}$ by $e^{A(x)} = x$ to get

$$\underbrace{xy' + y}_{\text{always } (e^{A(x)}y)'} = 3x^3 + 1$$

This becomes:

$$(xy)' = 3x^3 + 1$$

Integrate to get

$$xy = \int (3x^3 + 1)\, dx$$

We get

$$xy = \frac{3}{4}x^4 + x + C$$

Solve for y to get

$$y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$$

So, $y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$ solves

$xy' + y = 3x^3 + 1$ on $I = (0, \infty)$

<u>Ex:</u> Solve

$$\boxed{\begin{aligned} xy' + y &= 3x^3 + 1 \\ y(1) &= 2 \end{aligned}}$$

on $I = (0, \infty)$

We know from above that

$$y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$$

solves $xy' + y = 3x^3 + 1$.

Let's make $y(1) = 2$.

Want $\underbrace{\frac{3}{4}(1)^3 + 1 + \frac{c}{1}}_{y(1)} = 2$

Get: $\frac{7}{4} + c = 2$

So, $C = 2 - \frac{7}{4} = \frac{1}{4}$

$C = \frac{1}{4}$

The answer is

$$\boxed{y = \frac{3}{4}x^3 + 1 + \frac{1/4}{x}}$$