Math 2150-02

9/29/25

Topic 8 - Method of Undetermined Coefficients

We want to solve

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

where a_0, a_1, a_2 are constants

Ex:

$$y'' + 3y' + 2y = 2x^2$$

$$y'' - y' + y = 2\sin(3x)$$

Method:

Step 1: Find the general solution y_h to the homogeneous equation
$$a_2 y'' + a_1 y' + a_0 y = 0$$

TOPIC 7

Step 2: Find a particular solution y_p to
$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

TOPIC 8

Step 3: The general solution to $a_2 y'' + a_1 y' + a_0 y = b(x)$ is $y = y_h + y_p$

TOPIC 6

How do we guess y_p in step 2? Thats what this topic is about. Here's a table for how to guess.

$b(x)$	guess y_p
constant	A
degree one polynomial like $2x+1$ or $5x$	$Ax+B$
degree two polynomial like $5x^2$ or x^2-x+1	Ax^2+Bx+C
$\sin(kx)$	$A\sin(kx)+B\cos(kx)$

where k is a constant	
$\cos(kx)$ where k is a constant	$A\sin(kx) + B\cos(kx)$
exponential such as e^{kx} or $10e^{kx}$ k is a constant	Ae^{kx}
degree one poly times exponential such as xe^{kx} or $(2x-1)e^{kx}$ where k is a constant	$(Ax+B)e^{kx}$

Ex: Solve

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve the homogeneous equation

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

$$\underbrace{(r+1)}_{\substack{r+1=0 \\ r=-1}}\underbrace{(r+2)}_{\substack{r+2=0 \\ r=-2}} = 0$$

So, $r = -1, -2$.

Thus the general solution is

$$\boxed{y_h = c_1 e^{-x} + c_2 e^{-2x}}$$

<u>Step 2:</u> Find a particular solution y_p to

$$y'' + 3y' + 2y = \underbrace{2x^2}_{\text{degree two polynomial}}$$

We guess

$$y_p = Ax^2 + Bx + C$$

We plug this into the ODE and try to find A, B, C

that make y_p solve the ODE.

$\boxed{A, B, C \text{ are constants to be determined}}$

We have

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug this into $y'' + 3y' + 2y = 2x^2$ to get:

$$\underbrace{(2A)}_{y_p''} + 3\underbrace{(2Ax+B)}_{y_p'} + 2\underbrace{(Ax^2+Bx+C)}_{y_p} = 2x^2$$

We get:

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

$$\underbrace{\boxed{2A}}_{2}x^2 + \underbrace{\boxed{(6A+2B)}}_{0}x + \underbrace{\boxed{(2A+3B+2C)}}_{0} = 2x^2$$

We want

$$\boxed{\begin{aligned} 2A &= 2 \quad &(1)\\ 6A + 2B &= 0 \quad &(2)\\ 2A + 3B + 2C &= 0 \quad &(3)\end{aligned}}$$

(1) gives $\boxed{A = 1.}$

Plug $A=1$ into (2) to get:

$$6(1) + 2B = 0$$

So, $\boxed{B = -3}$.

Plug $A=1$, $B=-3$ into ③ to get:

$$2(1)+3(-3)+2C=0$$

Then, $\boxed{C = 7/2}$

Thus,

$$\boxed{\begin{aligned} y_p &= Ax^2+Bx+C \\ &= x^2-3x+7/2 \end{aligned}}$$

<u>Step 3:</u> The general solution to

$$y''+3y'+2y=2x^2$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1e^{-x}+c_2e^{-2x}+x^2-3x+7/2}$$

where c_1, c_2 are any constants

Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: We need to solve the homogeneous eqn.

$$y'' - y' + y = 0$$

The characteristic poly. is

$$r^2 - r + 1 = 0$$

The roots are:

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}\sqrt{-1}}{2}$$

roots

$\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$i = \sqrt{-1}$

$i^2 = -1$

$$= \frac{1 \pm \sqrt{3}\, i}{2}$$

$$= \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$$

$\alpha \pm \beta i$

$\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$

Then,

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

$$= c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

where c_1, c_2 are any constants.

Next time:

finish this one...