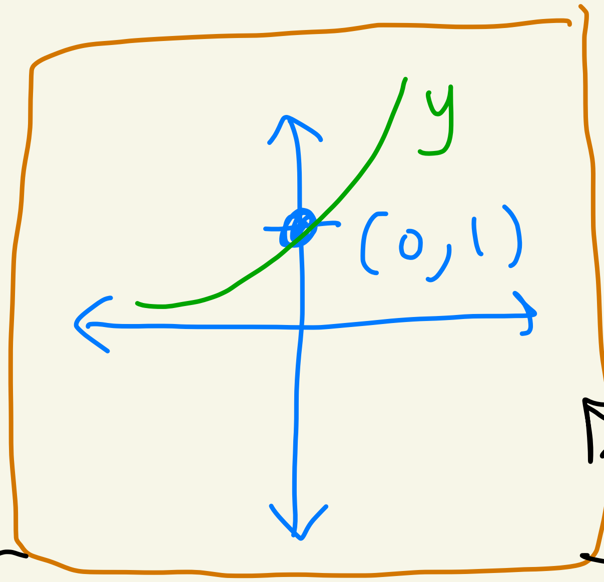
Math 2150
8/28/24

Topic 1 continued...

<u>Ex:</u> Let's try to find a solution to the initial-value problem

$$\boxed{\begin{gathered}\frac{dy}{dx} = y^2 \\ y(0) = 1\end{gathered}}$$



How to think about the condition $y(0)=1$

Last time we saw that

$$y = \frac{1}{c - x} \quad (\text{where } c \text{ is a constant})$$

solves $\frac{dy}{dx} = y^2$.

Let's try to solve $y(0)=1$ using $y = \frac{1}{c-x}$.

We would get

$$1 = y(0) = \frac{1}{c-0}$$

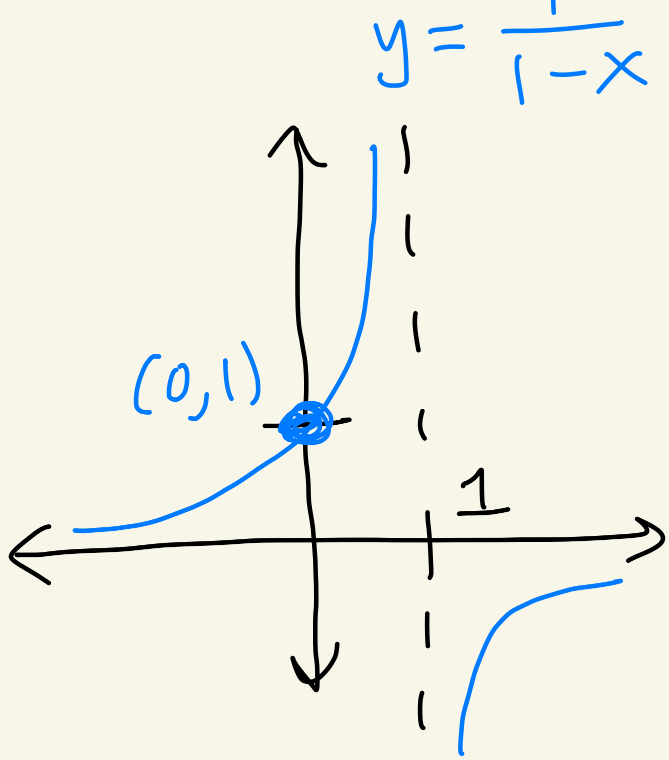
So,

$$1 = \frac{1}{c}$$

Thus, $c=1$.

So,

$$y = \frac{1}{1-x}$$



Solves the initial-value problem

Topic 2 - First order ODE theory

Let's discuss first order ODEs of the form

$$\boxed{\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$

equation you want y to solve

you want the graph of y to pass through (x_0, y_0)

Ex: Consider

$$\boxed{\begin{aligned} y' &= 2xy \\ y(0) &= 1 \end{aligned}}$$

$y' = f(x,y)$
$f(x,y) = 2xy$

Let

$$g(x) = e^{x^2}$$

Then,

$$g'(x) = \left(e^{x^2}\right)(2x) = 2x\underbrace{e^{x^2}}_{g}$$

So,

$$g'(x) = 2x\,g(x)$$

So, g solves $y' = 2xy$.

Also, $g(0) = e^{0^2} = e^0 = 1$.

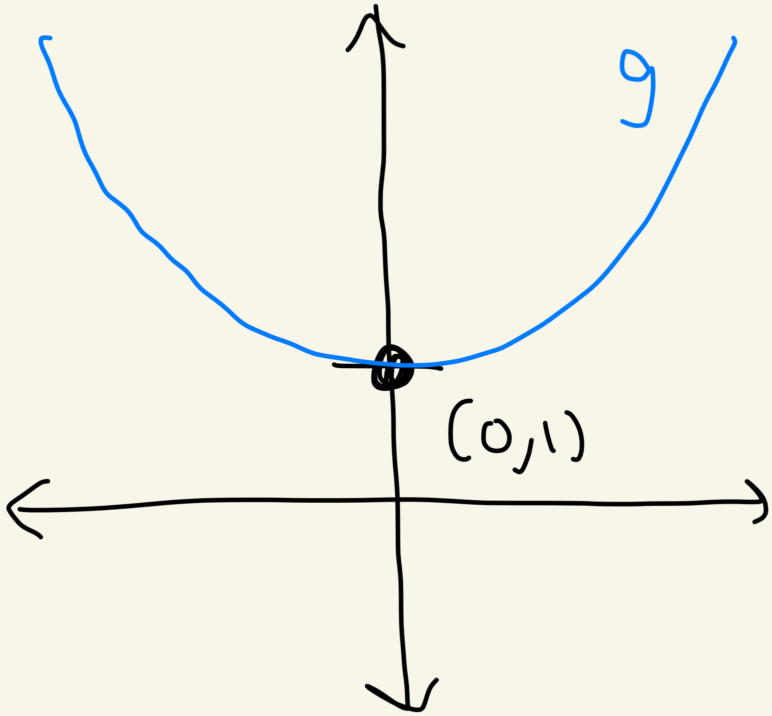
Thus,

$$\boxed{g(x) = e^{x^2}}$$

solves $\boxed{y' = 2xy,\ y(0) = 1.}$

It turns out that $g(x) = e^{x^2}$ is the only solution to

$$y' = 2xy$$
$$y(0) = 1$$



<u>Ex:</u> Consider the initial-value problem

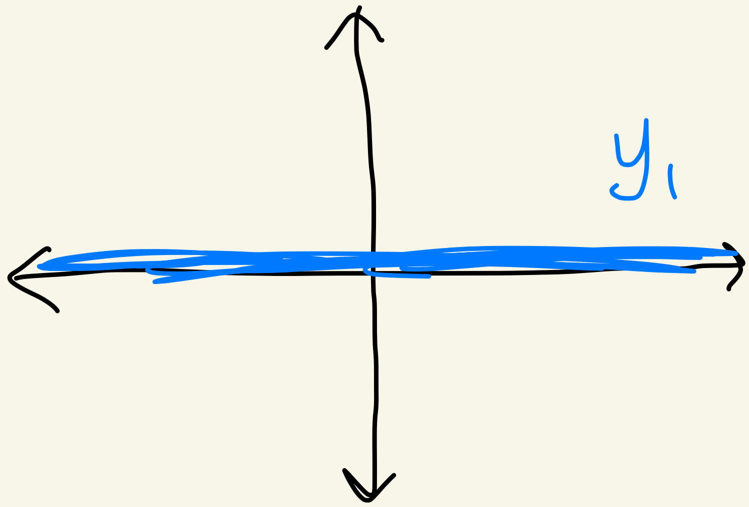
$$\boxed{\begin{aligned} \frac{dy}{dx} &= x y^{1/2} \\ y(0) &= 0 \end{aligned}}$$

Solution 1:

Let

$$y_1(x) = 0$$

for all x.



Then,

$$\frac{dy_1}{dx} = 0$$

Also,

$$x y^{1/2} = x \cdot 0^{1/2} = 0.$$

So,

$$\frac{dy_1}{\underbrace{dx}_{0}} = x \underbrace{y}_{0}^{1/2}.$$

Also, $y_1(0) = 0$.

So, y_1 solves the problem.

Solution 2:

Let $y_2(x) = \frac{1}{16}x^4$.

Then, $y_2'(x) = \frac{1}{4}x^3$

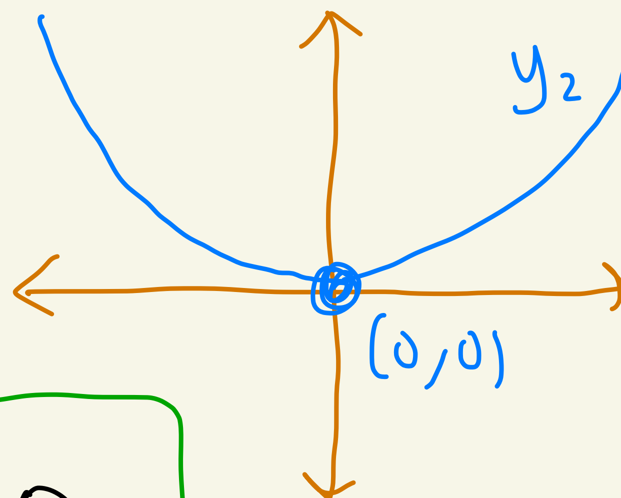
EQUAL

And,

$$x y_2^{1/2} = x\left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \frac{1}{4}x^2 = \frac{1}{4}x^3$$

So,

$$y_2' = x y_2^{1/2}$$



And

$$y_2(0) = \frac{1}{16}(0)^4 = 0.$$

Thus, y_2 solves

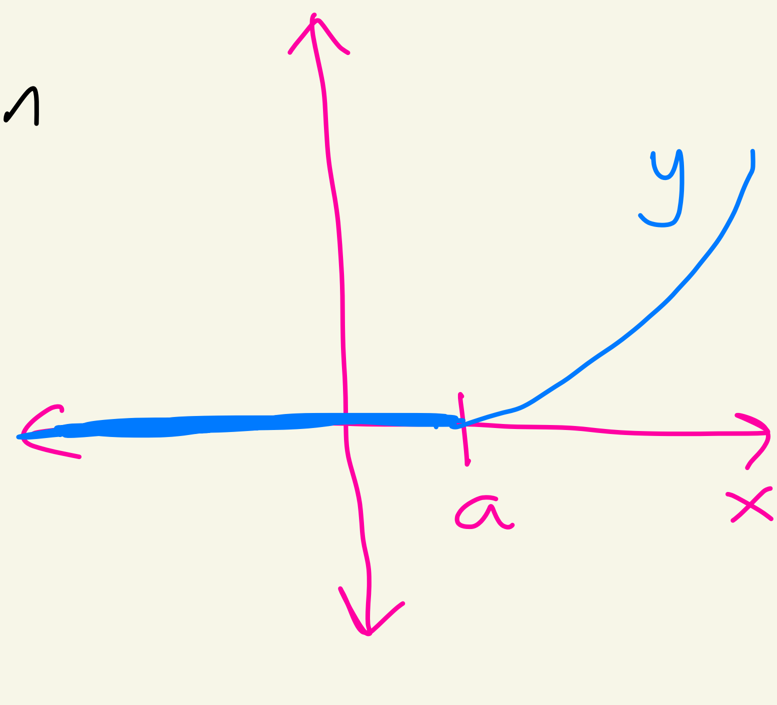
$$\boxed{\begin{array}{l} y' = x y^{1/2} \\ y(0) = 0 \end{array}}$$

More solutions:

There are an infinite number of solutions to this problem. For example, you can check that

$$y(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{(x^2 - a^2)^2}{16} & \text{if } x \geq a \end{cases}$$

is also a solution for any number $a \geq 0$.



Are there criteria that ensure that the problem

$$\frac{dy}{dx} = f(x,y)$$
$$y(x_0) = y_0$$

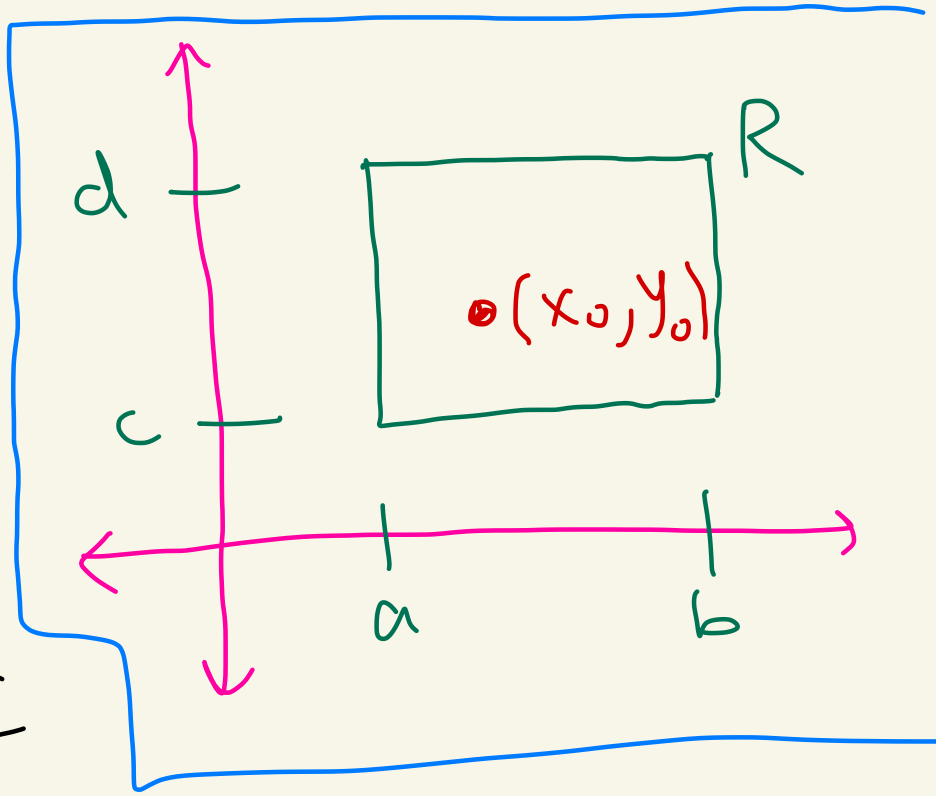
has a solution and its unique?

Answer: Yes

Theorem (due to Picard [1856-1941])

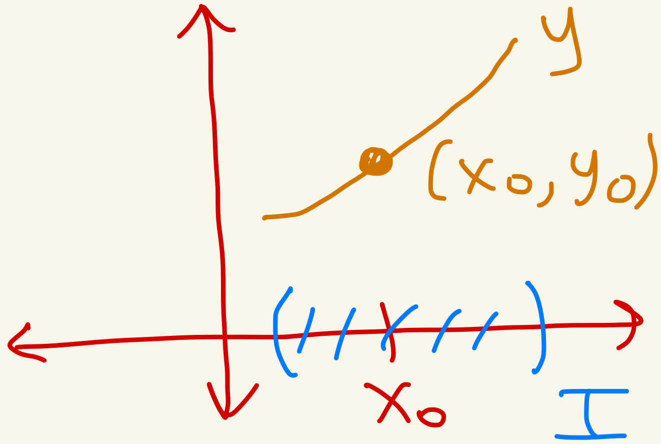
Let R be a rectangular region in the xy-plane defined by $a \le x \le b$ and $c \le y \le d$ [its ok if some of a,b,c,d are $\pm\infty$] that contains the point (x_0, y_0)

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in R, then there exists an interval I



centered at x_0 and a unique function $y(x)$ defined on I that satisfies

$$\boxed{\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$



Ex: Consider

$$\boxed{\begin{aligned} \frac{dy}{dx} &= 2xy \\ y(0) &= 1 \end{aligned}}$$

$\boxed{f(x,y) = 2xy}$

$\boxed{(x_0, y_0) = (0, 1)}$

We have

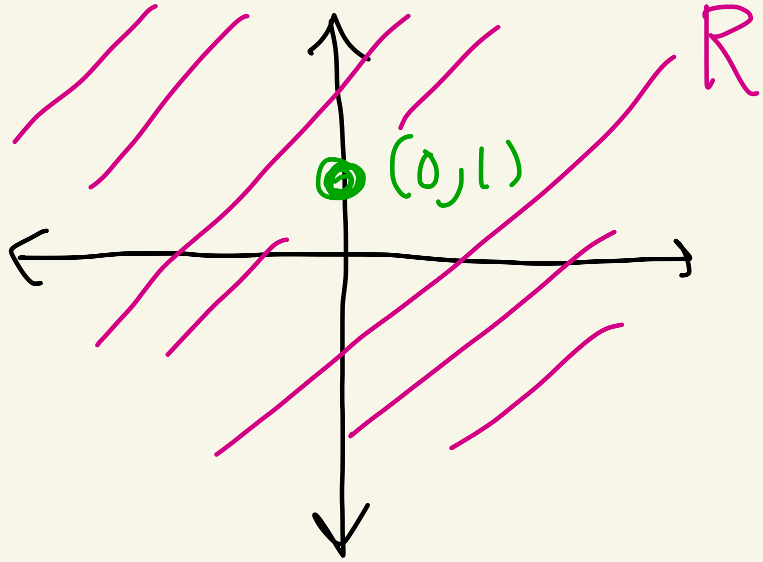
$$f(x,y) = 2xy$$

continuous everywhere!

$$\frac{\partial f}{\partial y} = 2x$$

continuous everywhere!

Let R be the entire xy-plane.
An infinite rectangle.



By Picard's theorem there is an interval I around $x_0 = 0$ and a unique solution $y(x)$ to the problem on that interval.

Here is the answer:

$$y(x) = e^{x^2}$$

$$I = (-\infty, \infty)$$