Math 2550-04
10/21/24

Ex: In $\mathbb{R}^3$, let

$\vec{a} = \langle -1, 0, 1 \rangle$, $\vec{b} = \langle -5, -3, 2 \rangle$, $\vec{c} = \langle 1, 1, 0 \rangle$

Notice that

$$\underbrace{\langle -5, -3, 2 \rangle = 2\langle -1, 0, 1 \rangle - 3\langle 1, 1, 0 \rangle}_{\vec{b} = 2\vec{a} - 3\vec{c}}$$

So, $\vec{b}$ is in the span of $\vec{a}$ and $\vec{c}$.

Thus, $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

Another way to write

$$\vec{b} = 2\vec{a} - 3\vec{c}$$

is

$$2 \cdot \vec{a} - 1 \cdot \vec{b} - 3\vec{c} = \vec{0}$$

<u>Fact:</u> The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are linearly independent if the only solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_r\vec{v}_r = \vec{0}$$

is $c_1 = 0, c_2 = 0, \ldots, c_r = 0$.

If there are more solutions then the vectors are linearly dependent.

<u>Ex:</u> Are $\vec{v}_1 = \langle 1, -2, 1 \rangle$, $\vec{v}_2 = \langle 1, 0, 1 \rangle$, $\vec{v}_3 = \langle 0, 1, 0 \rangle$ linearly dependent or linearly independent?

We want to solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

which gives

$$c_1\langle 1,-2,1\rangle + c_2\langle 1,0,1\rangle + c_3\langle 0,1,0\rangle = \langle 0,0,0\rangle$$

This gives

$$\langle c_1,-2c_1,c_1\rangle + \langle c_2,0,c_2\rangle + \langle 0,c_3,0\rangle = \langle 0,0,0\rangle$$

This becomes

$$\langle c_1+c_2,\ -2c_1+c_3,\ c_1+c_2\rangle = \langle 0,0,0\rangle$$

So,

$$\boxed{\begin{aligned} c_1 + c_2 \quad &= 0 \\ -2c_1 \quad + c_3 &= 0 \\ c_1 + c_2 \quad &= 0 \end{aligned}}$$

Let's solve.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} c_1 + c_2 \quad &= 0 \\ c_2 + \tfrac{1}{2}c_3 &= 0 \\ 0 &= 0 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

leading: c_1, c_2

free: c_3

We get:

$$\boxed{\begin{aligned} &① \ c_1 = -c_2 \\ &② \ c_2 = -\tfrac{1}{2} c_3 \\ &③ \ c_3 = t \end{aligned}}$$

Back substitute:

③ $c_3 = t$

② $c_2 = -\frac{1}{2} c_3 = -\frac{1}{2} t$

① $c_1 = -c_2 = -\left(-\frac{1}{2} t\right) = \frac{1}{2} t$

Plug this back into

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

to get

$$\left(\frac{1}{2} t\right) \vec{v}_1 - \left(\frac{1}{2} t\right) \vec{v}_2 + t\, v_3 = \vec{0}$$

for any t.

Plug in $t = 2$ for ex. to get

$$\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \vec{0}$$

So for example

$$\vec{v}_1 = \vec{v}_2 - 2\vec{v}_3$$

So the vectors are linearly dependent.

Ex: Let $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ in $\mathbb{R}^2$. Are $\vec{i}, \vec{j}$ lin. ind. or lin. dep.?

Let's solve

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

We get

$$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

This gives

$$\langle c_1, 0\rangle + \langle 0, c_2\rangle = \langle 0, 0\rangle$$

So,

$$\langle c_1, c_2\rangle = \langle 0, 0\rangle$$

Thus,

$$c_1 = 0 \quad \text{and} \quad c_2 = 0.$$

So, the only solution to

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

is $c_1 = 0, c_2 = 0$.

Thus, $\vec{i}, \vec{j}$ are linearly independent.

$\boxed{\text{Coordinate system theorem}}$

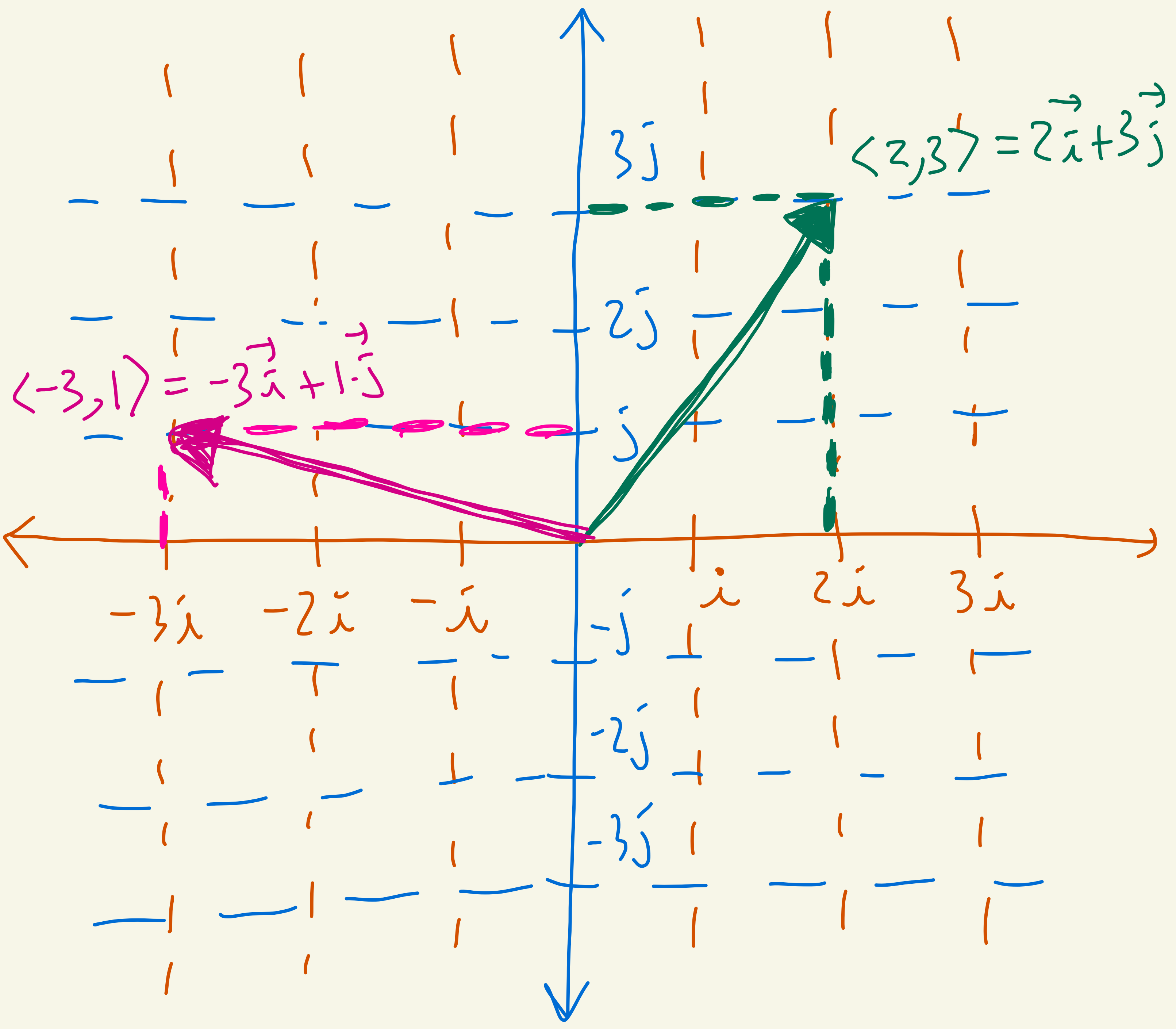
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n linearly independent vectors in $\mathbb{R}^n$. Then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ create a <u>coordinate system</u> for $\mathbb{R}^n$. That is, every vector $\vec{v}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

If we write $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ then we mean that we are fixing the ordering on the vectors and we are giving the coordinate system the name β. People also call the vectors in β a <u>basis</u>.

When $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ then $c_1, c_2, \dots, c_n$ are called the <u>coordinates</u> of $\vec{v}$ with respect to β and we write $[\vec{v}]_\beta = \langle c_1, c_2, \dots, c_n \rangle$

Ex: $\beta = [\vec{i}, \vec{j}]$ where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$. We have 2 lin. ind. vectors in $\mathbb{R}^2$. So, β is a coordinate system or basis for $\mathbb{R}^2$.



$\vec{v} = \langle 2, 3 \rangle$ $\qquad \beta = [\vec{i}, \vec{j}]$

$\vec{v} = 2\vec{i} + 3\vec{j}$

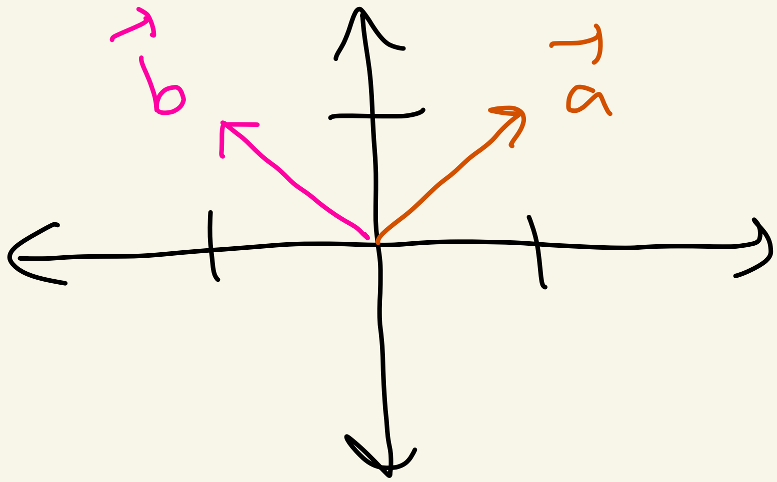
$[\vec{v}]_\beta = \langle 2, 3 \rangle$

$\vec{w} = \langle -3, 1 \rangle = -3\vec{i} + 1 \cdot \vec{j}$

$[\vec{w}]_\beta = \langle -3, 1 \rangle$

$\beta = [\vec{i}, \vec{j}]$ is called the <u>standard basis</u> (or <u>standard coordinate system</u>) for $\mathbb{R}^2$

Ex: Let $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$ be in $\mathbb{R}^2$. Let's show they are linearly independent.



We must solve

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

This gives

$$c_1 \langle 1, 1 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

This becomes

$$\langle c_1 - c_2, c_1 + c_2 \rangle = \langle 0, 0 \rangle$$

We get

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

We have

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

So,

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_2 &= 0 \quad (2) \end{aligned}}$$

no free variables

Thus,

(2) $c_2 = 0$

(1) $c_1 = c_2 = 0$

Thus, the only solution to

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$.

So, $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$ are linearly independent.