Math 2550-04
10/28/24

(Topic 6 continued...)

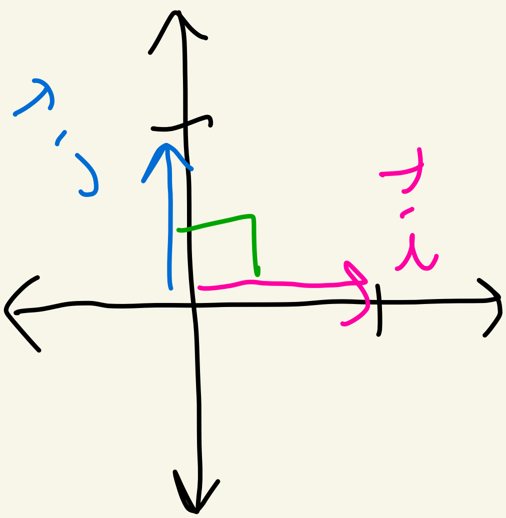
Def: Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be a basis/coordinate system for $\mathbb{R}^n$

- We say that β is orthogonal if every pair of vectors from β are orthogonal to each other, that is $\vec{v}_a \cdot \vec{v}_b = 0$ if $a \neq b$.

- We say that β is orthonormal if β is orthogonal and all the vectors in β have length 1.

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{i}, \vec{j}]$ where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$.

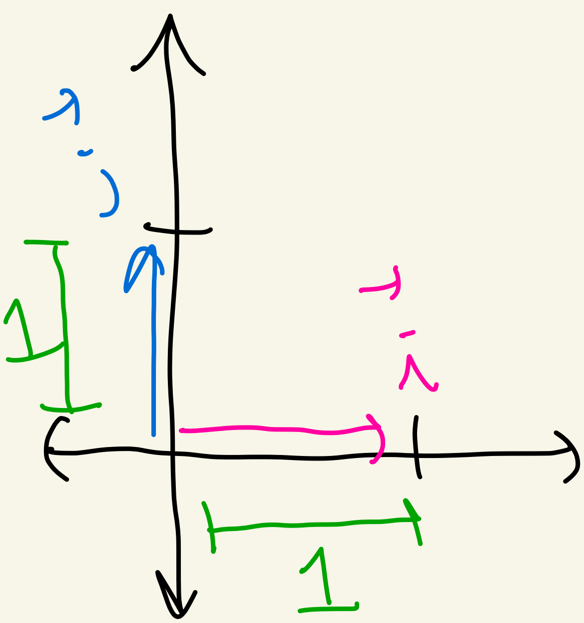
We saw before this is a basis.



We have

$$\vec{i} \cdot \vec{j} = 1 \cdot 0 + 0 \cdot 1 = 0$$

So, β is an orthogonal basis.



Also,

$$\|\vec{i}\| = \sqrt{1^2 + 0^2} = 1$$

$$\|\vec{j}\| = \sqrt{0^2 + 1^2} = 1$$

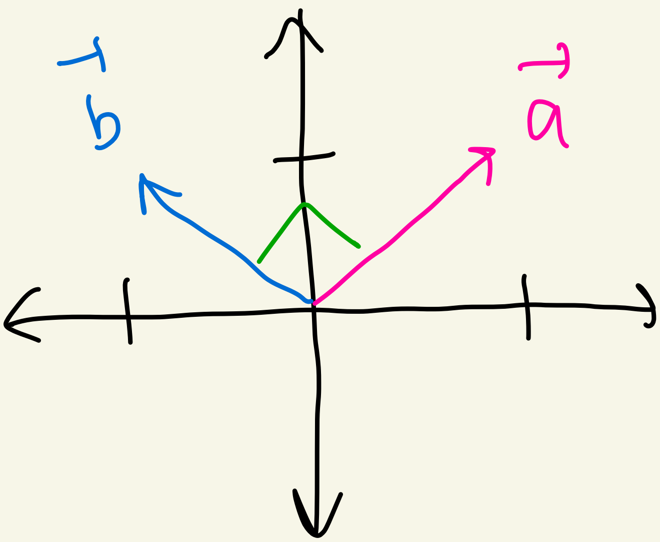
So, β is also orthonormal.

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$

Where $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$

We saw this is a basis previously.

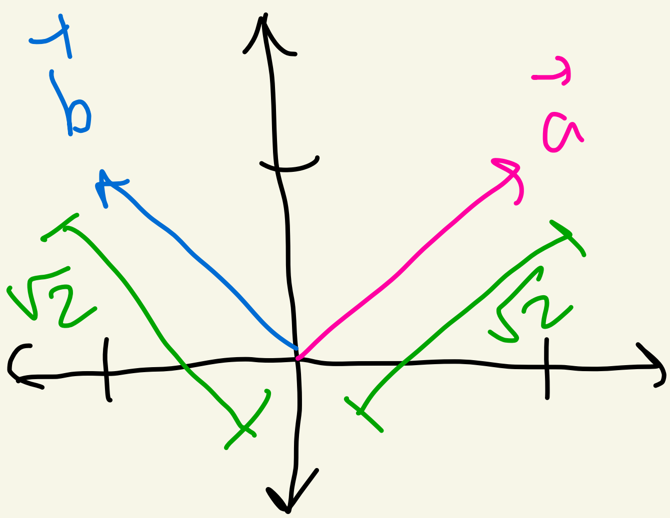
Is it orthogonal?



$\vec{a} \cdot \vec{b} = (1)(-1) + (1)(1)$
$= 0$

Yes, β is an orthogonal basis.

Is β orthonormal?



$||\vec{a}|| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1$

$||\vec{b}|| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \neq 1$

β is not orthonormal.

Ex: (HW 6 #3)

In $\mathbb{R}^2$, let $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle 1,0 \rangle$

- Let's show $\vec{a}, \vec{b}$ are linearly independent.

Need to solve

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

for c_1, c_2.

We get

$$c_1 \langle 1,1 \rangle + c_2 \langle 1,0 \rangle = \langle 0,0 \rangle$$

which is

$$\langle c_1, c_1 \rangle + \langle c_2, 0 \rangle = \langle 0,0 \rangle$$

$$\langle c_1 + c_2, c_1 \rangle = \langle 0,0 \rangle$$

So we get

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ c_1 &= 0 \quad (2) \end{aligned}}$$

(2) gives $c_1 = 0$. Plug into (2) to get $(0) + c_2 = 0$. So $c_2 = 0$.

Thus, the only solution to

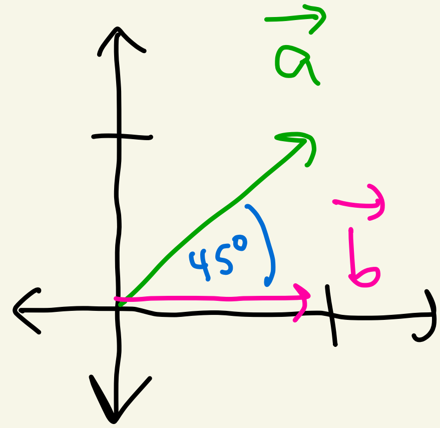
$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$.

So, $\vec{a}, \vec{b}$ are linearly independent.

Thus, $\beta = [\vec{a}, \vec{b}]$ is a basis/coordinate system for $\mathbb{R}^2$.

<u>Q:</u> Is $\beta = [\vec{a}, \vec{b}]$ an orthogonal basis?



Recall $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle 1, 0 \rangle$

We get

$$\vec{a} \cdot \vec{b} = (1)(1) + (1)(0) = 1 \neq 0.$$

So, β is <u>not</u> an orthogonal basis.

<u>Q:</u> Is β orthonormal?

No, because its not orthogonal.

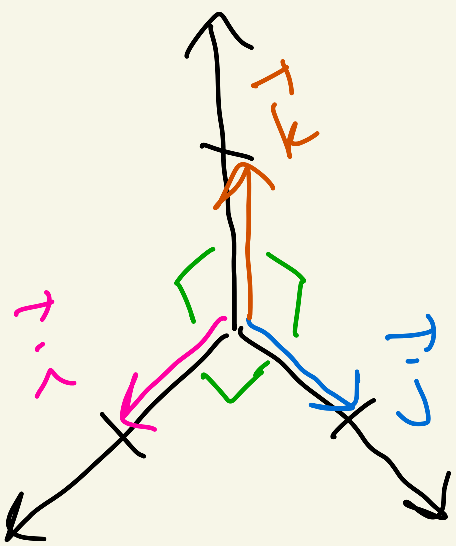
[orthonormal = (orthogonal) + (all vectors length 1)]

Ex: In $\mathbb{R}^3$, let $\beta = [\vec{i}, \vec{j}, \vec{k}]$

where $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$,
$\vec{k} = \langle 0, 0, 1 \rangle$.

We saw previously that β is a basis for $\mathbb{R}^3$.

Q: Is β an orthogonal basis?



We have

$$\vec{i} \cdot \vec{j} = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$$
$$\vec{i} \cdot \vec{k} = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$$
$$\vec{j} \cdot \vec{k} = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$$

All the dot products are 0.
So, β is an orthogonal basis.

Q: Is β an orthonormal basis?

β is orthogonal ✓

$\|\vec{i}\| = \sqrt{1^2 + 0^2 + 0^2} = 1$

$\|\vec{j}\| = \sqrt{0^2 + 1^2 + 0^2} = 1$

$\|\vec{k}\| = \sqrt{0^2 + 0^2 + 1^2} = 1$

So yes, β is an orthonormal basis.

Coordinate dot-product theorem

Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be a basis for $\mathbb{R}^n$.

Let $\vec{v}$ be a vector in $\mathbb{R}^n$.

- If β is an orthogonal basis, then

$$\vec{v} = \boxed{\frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}}\vec{v}_1 + \boxed{\frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}}\vec{v}_2 + \cdots + \boxed{\frac{\vec{v}\cdot\vec{v}_n}{\|\vec{v}_n\|^2}}\vec{v}_n$$

$\vec{v}$'s β-coordinates

- If β is an orthonormal basis, then

$$\vec{v} = \boxed{(\vec{v}\cdot\vec{v}_1)}\vec{v}_1 + \boxed{(\vec{v}\cdot\vec{v}_2)}\vec{v}_2 + \cdots + \boxed{(\vec{v}\cdot\vec{v}_n)}\vec{v}_n$$

$\vec{v}$'s β-coordinates

Ex: In $\mathbb{R}^2$, consider the orthonormal basis $\beta = [\vec{i}, \vec{j}]$ where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$.

Let $\vec{v} = \langle 9, -7 \rangle$.

Let's find $[\vec{v}]_\beta$.

The coordinate dot-product thm says

$$\vec{v} = (\vec{v} \cdot \vec{i})\vec{i} + (\vec{v} \cdot \vec{j})\vec{j}$$

$$= ((9)(1) + (-7)(0))\vec{i} + ((9)(0) + (-7)(1))\vec{j}$$

$$= 9\vec{i} - 7\vec{j}$$

So, $[\vec{v}]_\beta = \langle 9, -7 \rangle$

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$ where $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$.

We saw that β is an orthogonal basis.

Let $\vec{v} = \langle -8, 7 \rangle$.

Let's find $[\vec{v}]_\beta$.

The coordinate dot-product thm says:

$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \right) \vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$$

$$= \left(\frac{(-8)(1) + (7)(1)}{\left(\sqrt{1^2 + 1^2}\right)^2} \right) \vec{a} + \left(\frac{(-8)(-1) + (7)(1)}{\left(\sqrt{(-1)^2 + 1^2}\right)^2} \right) \vec{b}$$

$$= -\frac{1}{2} \vec{a} + \frac{15}{2} \vec{b}$$

Check:

$$-\frac{1}{2}\langle 1,1\rangle + \frac{15}{2}\langle -1,1\rangle$$

$$= \left\langle -\frac{1}{2}-\frac{15}{2},\ -\frac{1}{2}+\frac{15}{2}\right\rangle = \langle -8, 7\rangle$$

So, $[\vec{v}]_\beta = \underbrace{\left\langle -\frac{1}{2}, \frac{15}{2}\right\rangle}_{\vec{v} = -\frac{1}{2}\vec{a} + \frac{15}{2}\vec{b}}$.

Note: To turn an orthogonal basis into an orthonormal basis, just divide each vector by its length.

Ex:

orthogonal

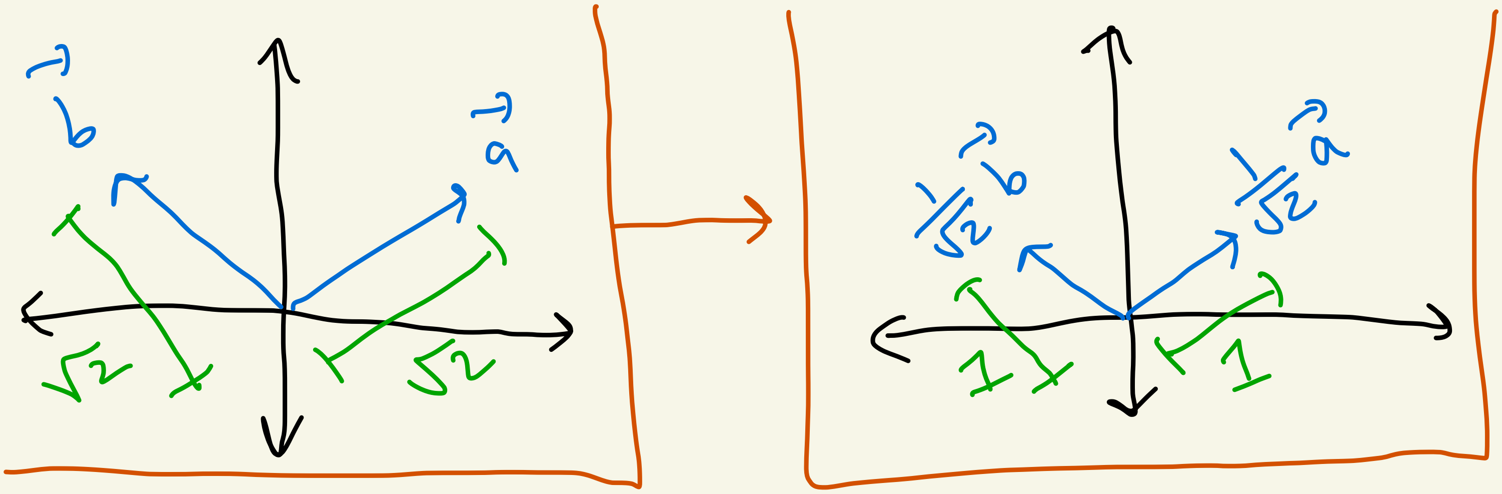
$\vec{a} = \langle 1, 1\rangle$

$\vec{b} = \langle -1, 1\rangle$

orthonormal

$$\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{2}}\vec{a} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$\frac{1}{\|\vec{b}\|}\vec{b} = \frac{1}{\sqrt{2}}\vec{b} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$



$\vec{b}$
$\vec{a}$
$\sqrt{2}$
$\sqrt{2}$
$\frac{1}{\sqrt{2}}\vec{b}$
$\frac{1}{\sqrt{2}}\vec{a}$
1
1