Math 2550-04
11/18/24

Topic 8 - Linear Transformations and Eigenvalues

<u>Def</u>: Given two sets A and B a <u>function</u> f from A to B is a rule that assigns to each x in A a unique element $f(x)$ in B.

We write $f: A \to B$.

where the outputs live

name of function

domain or input to function

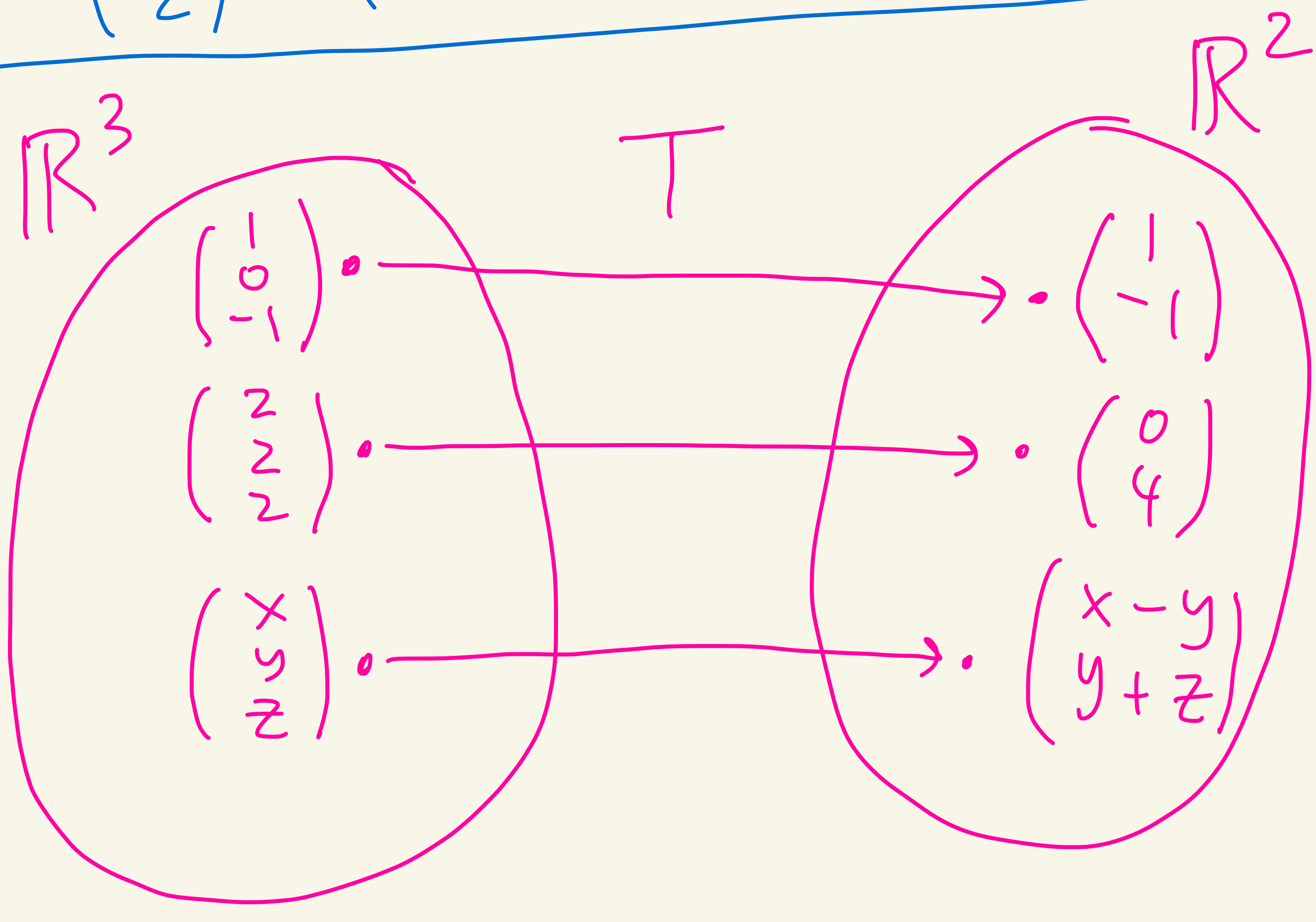
Ex: Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$

where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \end{pmatrix}$

Some example calculations are:

$$T\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 2+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$



Note that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{v}}$$

If $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ then

$$T(\vec{v}) = A\vec{v}$$

Def: A linear transformation is a function $T: \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{v}) = A\vec{v}$ and A is an $m \times n$ matrix.

Ex: $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

above is a linear transformation

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

is a linear transformation

Note that

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+3y \\ 5y \end{pmatrix}$$

For example

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5(1)+3(2) \\ 5(2) \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then T is a linear transformation if and only if the following two properties hold for any vectors $\vec{v}, \vec{w}$ and scalar α:

1. $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$
2. $T(\alpha\vec{v}) = \alpha T(\vec{v})$

Why are ① & ② true if $T(\vec{v}) = A\vec{v}$ is a linear transformation?

① $T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w})$

$= A\vec{v} + A\vec{w}$

$= T(\vec{v}) + T(\vec{w})$

matrix property

② $T(\alpha\vec{v}) = A(\alpha\vec{v}) = \alpha(A\vec{v})$

$= \alpha T(\vec{v})$

<u>Def:</u> Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $T(\vec{v}) = A\vec{v}$ where A is an $n \times n$ matrix.

Suppose that $\vec{v}$ is a vector in $\mathbb{R}^n$ with $\vec{v} \neq \vec{0}$ and $T(\vec{v}) = \lambda \vec{v}$ where λ is a scalar/number. Then λ is called an <u>eigenvalue</u> of T and $\vec{v}$ is called an <u>eigenvector</u> of T corresponding to λ.

Given an eigenvalue λ of T define the <u>eigenspace</u> of λ to be

$$E_\lambda(T) = \underbrace{\{ \vec{w} \mid T(\vec{w}) = \lambda \vec{w} \}}$$

this set consists of all eigenvectors corresponding to λ and also the zero vector $\vec{0}$
this will make $E_\lambda(T)$ a subspace

Ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then, T is a linear transformation and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x - 9y \\ 4x - 2y \end{pmatrix}$$

Note that

$$T\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}10\cdot 3 - 9\cdot 2\\ 4\cdot 3 - 2\cdot 2\end{pmatrix}$$

$$= \begin{pmatrix}12\\8\end{pmatrix}$$

$$= 4\begin{pmatrix}3\\2\end{pmatrix}$$

Thus, $\underbrace{T\begin{pmatrix}3\\2\end{pmatrix} = 4\cdot\begin{pmatrix}3\\2\end{pmatrix}}_{T(\vec{v}) = \lambda\vec{v}}$

So, $\lambda = 4$ is an eigenvalue of T and $\vec{v} = \begin{pmatrix}3\\2\end{pmatrix}$ is a corresponding eigenvector.

So, $E_4(T) = \left\{\begin{pmatrix}3\\2\end{pmatrix}, \dots\right\}$

more vectors

$$\boxed{\text{How do we find the eigenvalues of a linear transformation } T \text{ where } T(\vec{v}) = A\vec{v} \text{ and } A \text{ is an } n\times n \text{ matrix}}$$

Suppose $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda\vec{v}$.

Then $A\vec{v} - \lambda\vec{v} = \vec{0}$

$\boxed{I_n\vec{v} = \vec{v}}$

So $(A - \lambda I_n)\vec{v} = \vec{0}$

The only way this can happen is if $(A - \lambda I_n)$ has no inverse.

Why? Because if $(A-\lambda I_n)^{-1}$ existed then we get

$$\underbrace{(A-\lambda I_n)^{-1}(A-\lambda I_n)\vec{v}}_{\vec{v}} = \underbrace{(A-\lambda I_n)^{-1}\vec{0}}_{\vec{0}}$$

Then, $\vec{v} = \vec{0}$

But $\vec{v} \neq \vec{0}$.

So, $A - \lambda I_n$ has no inverse

and $\det(A - \lambda I_n) = 0$.

<u>Summary:</u> The eigenvalues of T or A are the λ that satisfy

$$\underbrace{\det(A - \lambda I_n)}_{\text{characteristic polynomial of } A} = 0$$