Math 2550-04
11/20/24

(Topic 8 continued...)

Recap from last time:

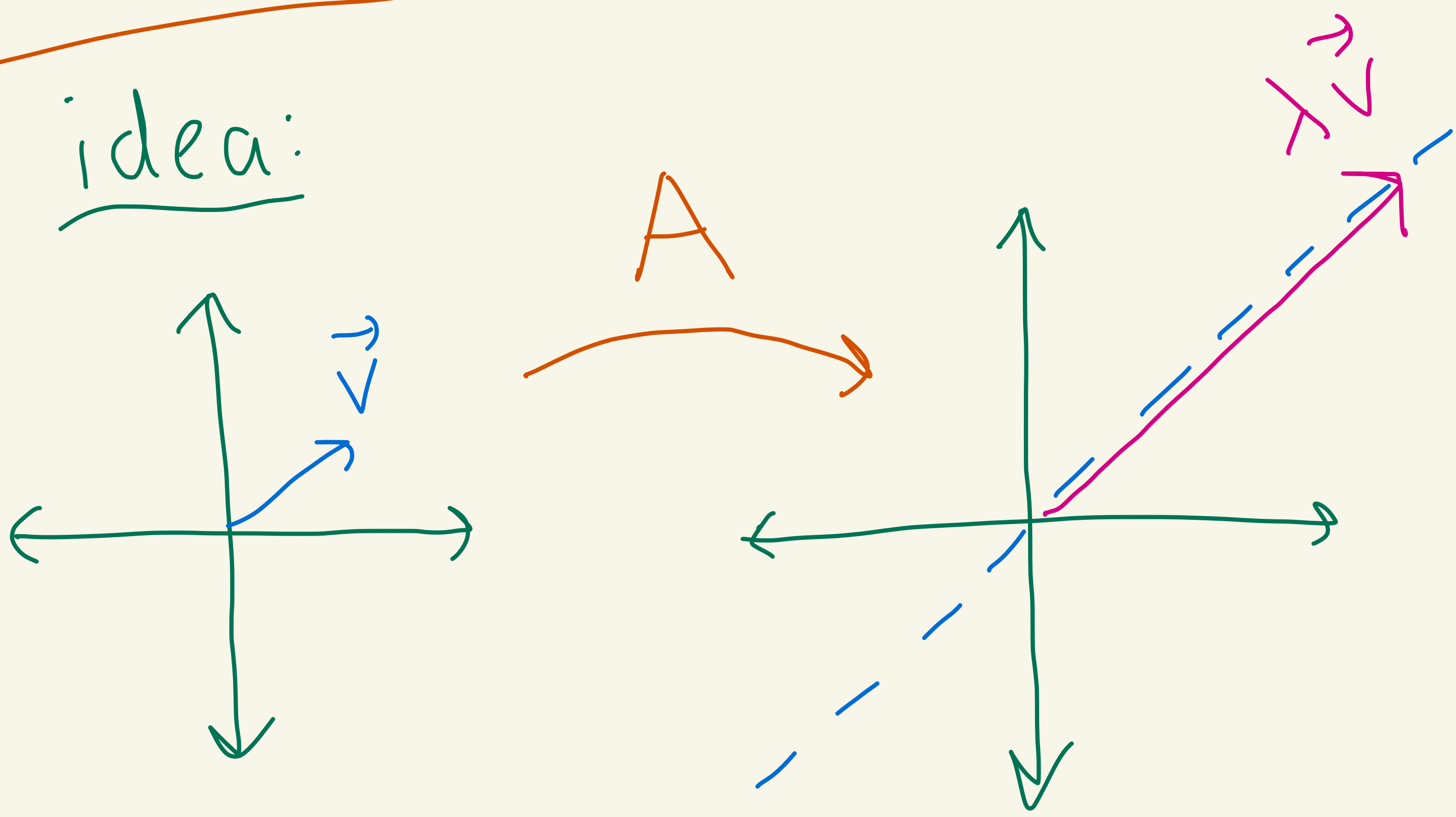
$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v} \neq \vec{0}$$

λ = eigenvalue

$\vec{v}$ = eigenvector

idea:



The eigenvalues λ are the roots of $\det(A-\lambda I)=0$

<u>Ex:</u> Let $A=\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Lets find the eigenvalues.

We get

$$\det(A-\lambda I)$$

$$= \det\left(\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}}_{A} - \lambda \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\right)$$

$$= \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) - (0)(8)$$

$$= (3-\lambda)(-1-\lambda)$$

When is $\det(A-\lambda I)=0$?

When $(3-\lambda)(-1-\lambda)=0$.

This when $\lambda = 3, -1$.

$$\boxed{\text{The eigenvalues of } A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \text{ are } \lambda = 3, -1}$$

Let's find the eigenvectors!

Let's start with $\lambda = -1$.

We need to solve $\boxed{\underbrace{A\vec{v} = -\vec{v}}_{A\vec{v} = \lambda \vec{v}}}$

Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Need to solve $\boxed{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}}$

multiply to get $\begin{pmatrix} 3x + 0y \\ 8x - y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

This is $\boxed{\begin{aligned} 3x &= -x \\ 8x - y &= -y \end{aligned}}$

This is $\boxed{\begin{aligned} 4x &= 0 \\ 8x &= 0 \end{aligned}}$

$$\left(\begin{array}{cc|c} 4 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-8R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\boxed{\begin{aligned} x &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: x

free: y

$$\boxed{\underline{\text{Solution:}} \quad \begin{aligned} x &= 0 \\ y &= t \end{aligned}}$$

<u>Summary:</u> Solutions to

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}}_{A\vec{v} = -\vec{v}}$$

are $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, the eigenvectors for $\lambda = -1$ all have the form $t\begin{pmatrix}0\\1\end{pmatrix}$.

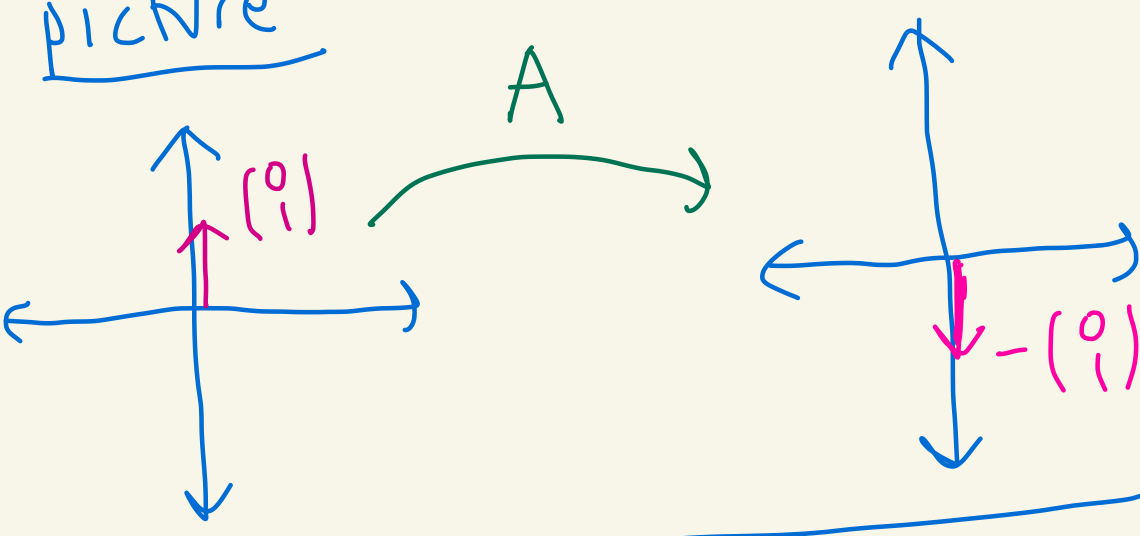
Some example eigenvectors for $\lambda = -1$ are: $\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{t=1}, \underbrace{\begin{pmatrix}0\\2\end{pmatrix}}_{t=2}, \underbrace{\begin{pmatrix}0\\-5\end{pmatrix}}_{t=-5}, \ldots$

So, the eigenspace $E_{-1}(A)$ is

$$E_{-1}(A) = \left\{ t\begin{pmatrix}0\\1\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\-5\end{pmatrix}, \ldots \right\}$$

picture



$$\boxed{\begin{aligned} A\vec{v} &= -\vec{v} \\ \vec{v} &= t\begin{pmatrix}0\\1\end{pmatrix} \end{aligned}}$$

What about $\lambda = 3$?

We need to solve $A\vec{v} = 3\vec{v}$.

This is $\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{v}} = \underbrace{3\begin{pmatrix} x \\ y \end{pmatrix}}_{3\vec{v}}$

We get $\begin{pmatrix} 3x + 0y \\ 8x - y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

This is $\boxed{\begin{aligned} 3x &= 3x \\ 8x - y &= 3y \end{aligned}}$

This is $\boxed{\begin{aligned} 0 &= 0 \\ 8x - 4y &= 0 \end{aligned}}$

This is $\downarrow$

$$\boxed{\begin{aligned} x - \tfrac{1}{2}y &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: x
free: y

Solution: $y = t$

$x = \frac{1}{2}y = \frac{1}{2}t$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

The eigenvectors corresponding to $\lambda = 3$ are of the form

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

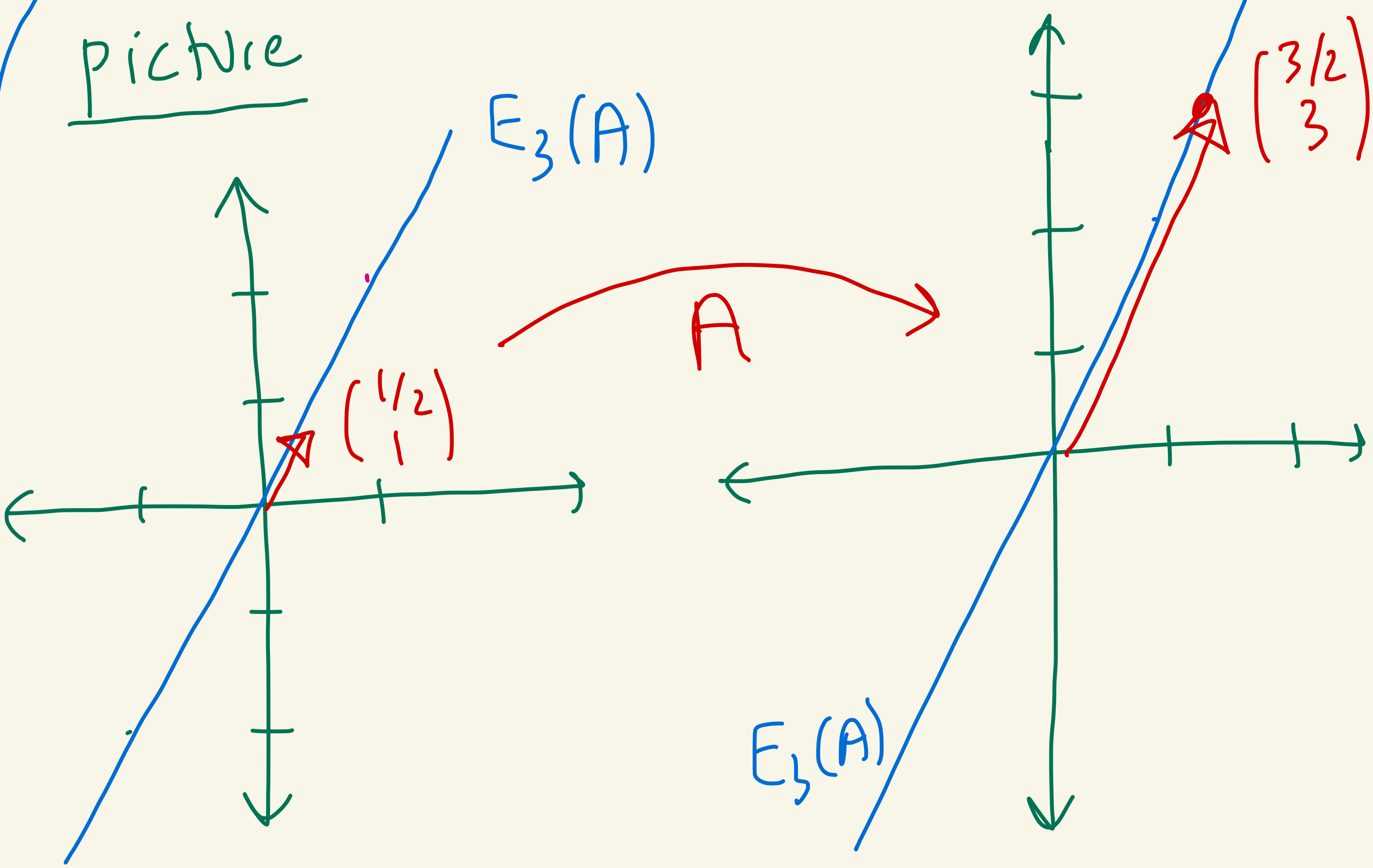
these solve $A\vec{v} = 3\vec{v}$

So,

$$E_3(A) = \left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \underline{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \begin{pmatrix} -5 \\ -10 \end{pmatrix}, \dots \right\}$$

eigenspace for $\lambda = 3$ $\quad t=1 \quad t=2 \quad t=-10$

picture



A stretches the vectors in $E_3(A)$ by $\lambda = 3$