2550-04

12-4-24

Test 2

② $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

Use A^{-1} to solve

$$\begin{aligned} x + y &= 2 \\ 2x + 3y &= 3 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

multiply by A^{-1} to get

$$\cancel{\underbrace{\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}}_{A}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \cdot 2 - 1 \cdot 3 \\ -2 \cdot 2 + 1 \cdot 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\boxed{\text{Answer: } x = 3, y = -1}$$

Test 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

3

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$$

$$= 0 + 1 \cdot \underbrace{\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix}}_{\begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}} - 1 \cdot \underbrace{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}}_{\begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}}$$

$$= \left[1 \cdot 3 - 1 \cdot 4\right] - \left[1 \cdot 2 - 1 \cdot 3\right]$$

$$= \left[-1\right] - \left[-1\right] = 0$$

Thus, $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$ has no inverse since $\det(A) = 0$

$$\boxed{\text{Test 2}}$$

(4) Show that

$\vec{v} = \langle 1, 0 \rangle$, $\vec{u} = \langle 1, 3 \rangle$, $\vec{w} = \langle 1, 1 \rangle$

are linearly dependent and write one of them as a linear combination of the others.

$$c_1 \vec{V} + c_2 \vec{u} + c_3 \vec{w} = \vec{0}$$

$$c_1 \langle 1, 0 \rangle + c_2 \langle 1, 3 \rangle + c_3 \langle 1, 1 \rangle = \langle 0, 0 \rangle$$

$$\langle c_1, 0 \rangle + \langle c_2, 3c_2 \rangle + \langle c_3, c_3 \rangle = \langle 0, 0 \rangle$$

$$\langle c_1 + c_2 + c_3, 3c_2 + c_3 \rangle = \langle 0, 0 \rangle$$

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 3c_2 + c_3 &= 0 \end{aligned}} \xrightarrow{\frac{1}{3}R_2 \to R_2} \boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + \tfrac{1}{3}c_3 &= 0 \end{aligned}}$$

free: c_3

leading: c_1, c_2

$$c_3 = t$$

$$c_2 = -\frac{1}{3}c_3 = -\frac{1}{3}t$$

$$c_1 = -c_2 - c_3$$

$$= \frac{1}{3}t - t = -\frac{2}{3}t$$

Plug this back into

$$c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}$$

to get

$$\left(-\frac{2}{3}t\right)\vec{v} + \left(-\frac{1}{3}t\right)\vec{u} + t\vec{w} = \vec{0}$$

Plug in $t = 3$ to get:

$$\boxed{-2\vec{v} - \vec{u} + 3\vec{w} = \vec{0}}$$

So $\vec{v}, \vec{u}, \vec{w}$ are lin. dep.

$$\boxed{\vec{u} = -2\vec{v} + 3\vec{w}}$$

$\vec{u}$ as a lin. combo of $\vec{v}$ and $\vec{w}$

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(5) (a) Show that

$\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 2, -1 \rangle$

are linearly independent.

Consider

$$c_1\vec{a} + c_2\vec{b} = \vec{0}$$

$$c_1\langle 1, 2\rangle + c_2\langle 2, -1\rangle = \langle 0, 0\rangle$$

$$\langle c_1, 2c_1\rangle + \langle 2c_2, -c_2\rangle = \langle 0, 0\rangle$$

$$\langle c_1 + 2c_2,\ 2c_1 - c_2 \rangle = \langle 0, 0 \rangle$$

So,

$$\boxed{\begin{aligned} c_1 + 2c_2 &= 0 \\ 2c_1 - c_2 &= 0 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} c_1 + 2c_2 &= 0 \\ c_2 &= 0 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

So,

(2) $c_2 = 0$

(1) $c_1 = -2c_2 = -2(0) = 0$

We have shown that the only solutions to

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

are $c_1 = 0$, $c_2 = 0$.

Thus, $\vec{a}, \vec{b}$ are linearly independent.

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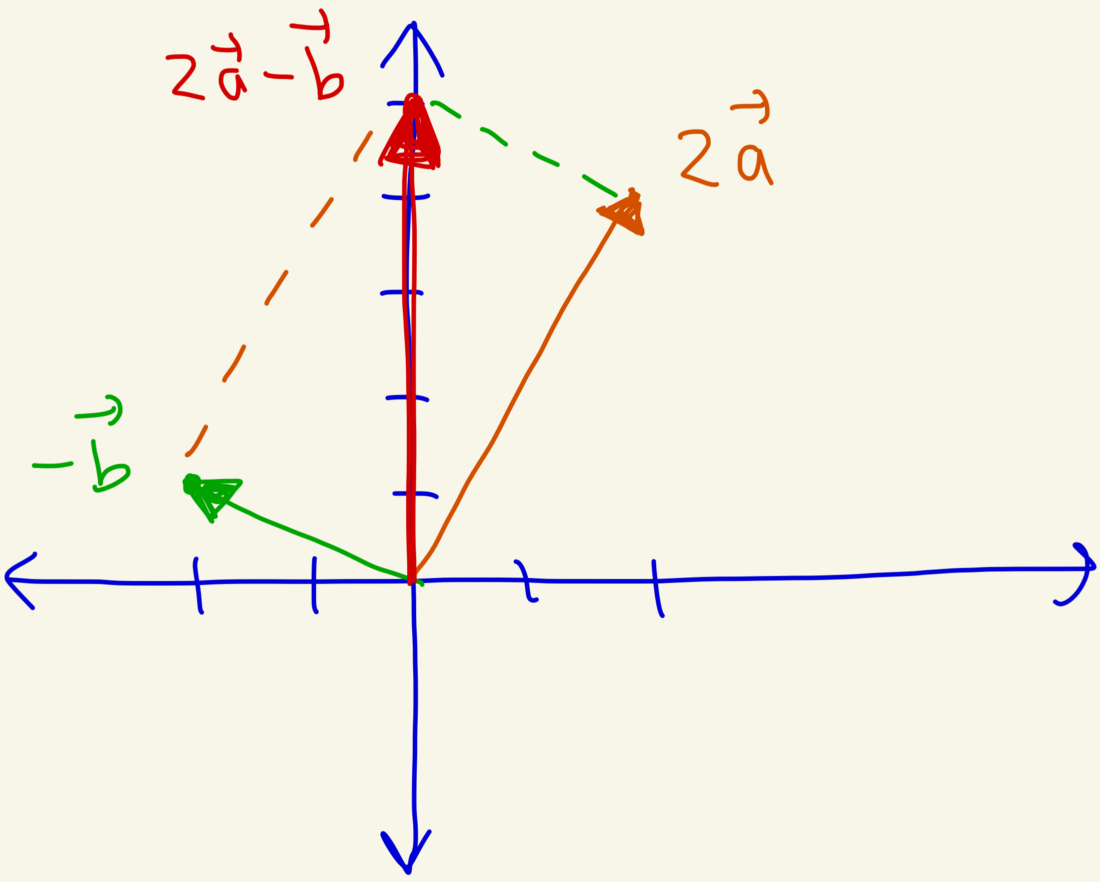
5 (b) $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 2, -1 \rangle$

Draw $2\vec{a}$, $-\vec{b}$, $2\vec{a} - \vec{b}$ and the parallelogram they make.



$2\vec{a} = \langle 2, 4 \rangle$

$-\vec{b} = \langle -2, 1 \rangle$

$2\vec{a} - \vec{b} = \langle 0, 5 \rangle$

5 (c) $\beta = [\vec{a}, \vec{b}]$, $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 2, -1 \rangle$

From (a) β is a basis.

Q1: Is β an orthogonal basis?

Q2: Is β an orthonormal basis?

$\boxed{\text{Q1}}$

$$\vec{a} \cdot \vec{b} = \langle 1, 2 \rangle \cdot \langle 2, -1 \rangle$$
$$= (1)(2) + (2)(-1)$$
$$= 0$$

So, β is an orthogonal basis

$\boxed{\text{Q2}}$ β orthogonal ✓

$$\|\vec{a}\| = \sqrt{(1)^2 + (2)^2} = \sqrt{5} \neq 1$$
$$\|\vec{b}\| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5} \neq 1$$

Need both $\|\vec{a}\| = 1$ and $\|\vec{b}\| = 1$

So, β is not orthonormal.

(5) (d) $\beta = [\vec{a}, \vec{b}]$, $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 2, -1 \rangle$

Use coordinate-dot product theorem to find $\underbrace{[\vec{v}]_\beta}_{\beta\text{-coordinates of } \vec{v}}$ where $\vec{v} = \langle 1, 1 \rangle$

[You can use the CDPT when β is orthogonal or orthonormal. We have β is orthogonal.]

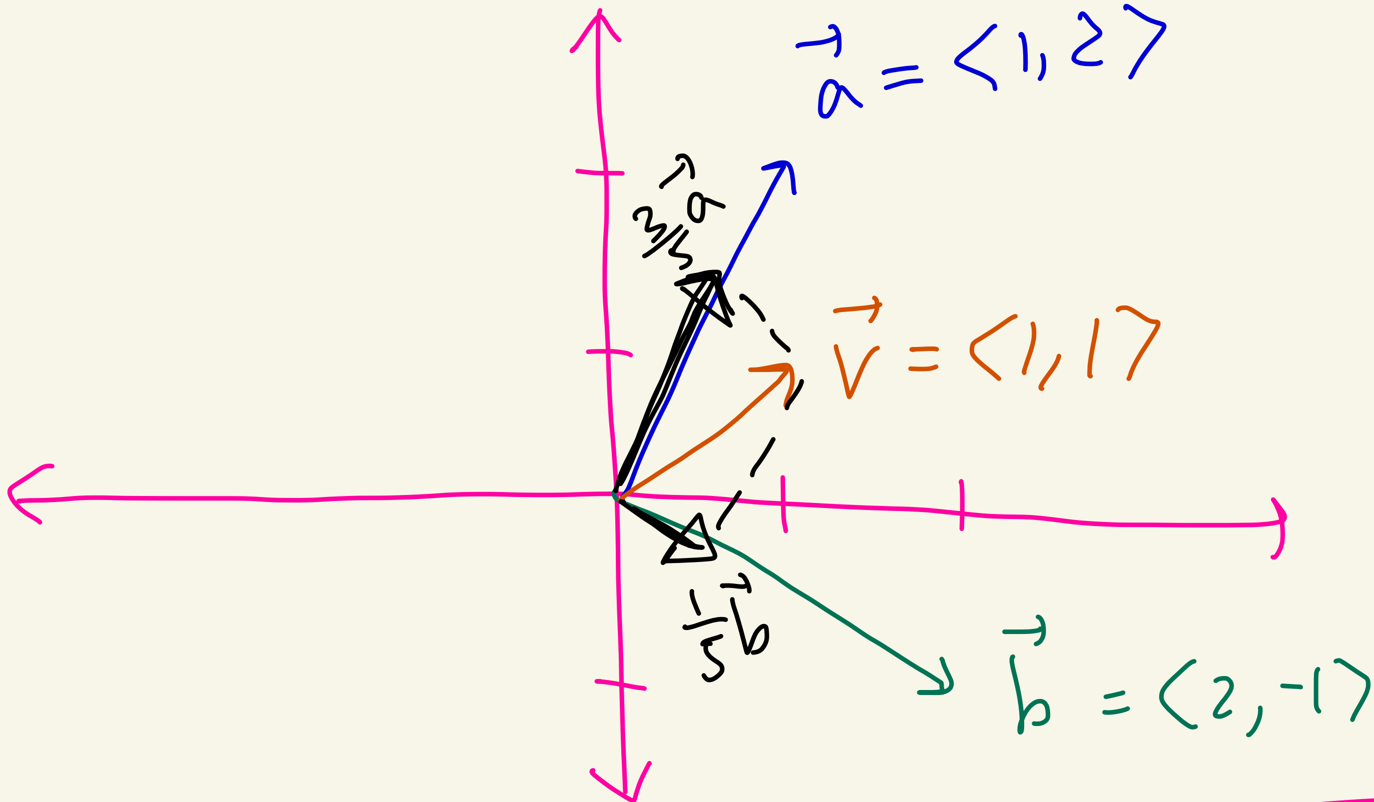
$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \right) \vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$$

$$= \left(\frac{\langle 1, 1 \rangle \cdot \langle 1, 2 \rangle}{(\sqrt{5})^2} \right) \vec{a} + \left(\frac{\langle 1, 1 \rangle \cdot \langle 2, -1 \rangle}{(\sqrt{5})^2} \right) \vec{b}$$

$$= \left(\frac{1 + 2}{5} \right) \vec{a} + \left(\frac{2 - 1}{5} \right) \vec{b}$$

$$= \frac{3}{5}\vec{a} + \frac{1}{5}\vec{b}$$

So, $[\vec{v}]_\beta = \langle \frac{3}{5}, \frac{1}{5} \rangle$



5 (e) (modified)

If $[\vec{v}]_\beta = \langle 2, 3 \rangle$ what is $\vec{v}$?

So, $\vec{v} = 2\vec{a} + 3\vec{b} = 2\langle 1, 2 \rangle + 3\langle 2, -1 \rangle$

$$= \langle 8, 1 \rangle$$