Math 2550-04
8/26/24

Topic 1 - Vectors

<u>Def</u>: Let $n \geq 1$ be an integer.

[So n can be $1, 2, 3, 4, \ldots$]

An <u>n-dimensional real vector</u> is a list of n real numbers. We use brackets < and > to denote vectors and commas to separate the numbers.

We use an arrow over a variable to denote a vector such as $\vec{v}$.

Ex: Some 2-dimensional vectors are:

$\langle \pi, e \rangle$

$\langle 1, -1 \rangle$

Ex: Some 3-dim. vectors:

$\langle 1, 2.5, -10 \rangle$

$\langle 0, 0, 0 \rangle$

Ex: A 5-dim vector:

$\langle -1, 0, 2, \frac{1}{2}, \pi \rangle$

Ex: A 10-dim. vector:

$\langle 2, \frac{1}{2}, 3, 4, -5, \pi, 1, 1, 1, 1 \rangle$

Order matters for vectors:

$$\langle 1,2,3\rangle \neq \langle 2,3,1\rangle$$

<u>Def:</u> We write $\mathbb{R}^n$ for the set of all n-dimensional real vectors.

So,

$$\mathbb{R}^n = \{\langle a_1, a_2, \ldots, a_n\rangle \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

<u>Ex:</u>

$$\mathbb{R}^2 = \{\langle a_1, a_2\rangle \mid a_1, a_2 \in \mathbb{R}\}$$

$$= \{\underbrace{\langle 0,0\rangle}_{a_1=0,\ a_2=0}, \underbrace{\langle 58,-58\rangle}_{\substack{a_1=58\\ a_2=-58}},$$

$$\langle \pi, 2.367 \rangle, \ldots \}$$

$a_1 = \pi$

$a_2 = 2.367$

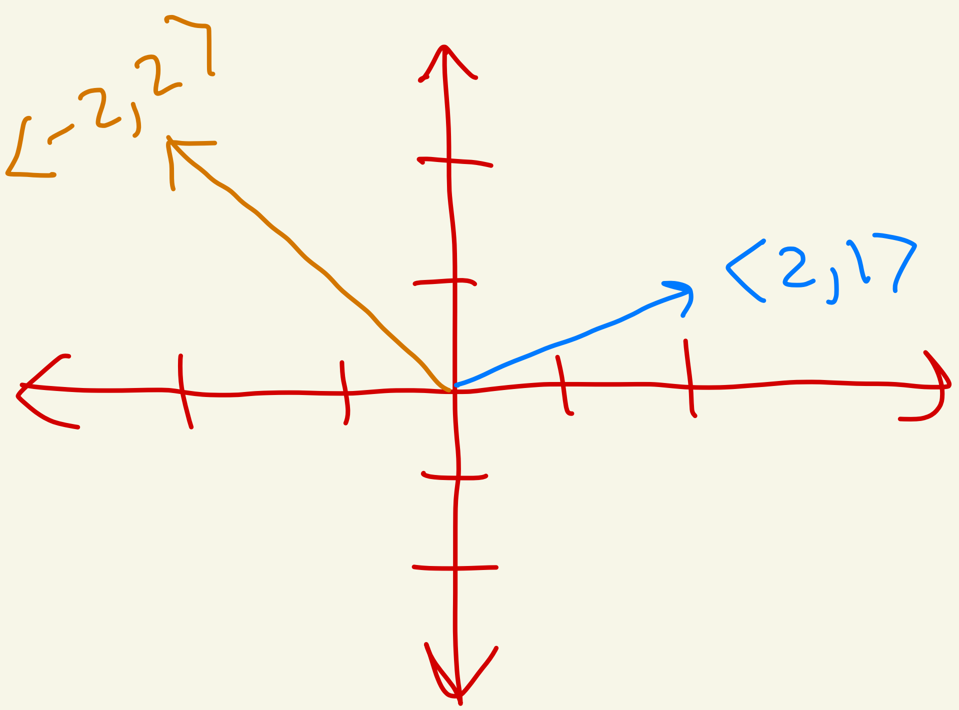
infinitely many more

$\mathbb{R}^2$

$\langle 0, 0 \rangle$ $\langle 58, -58 \rangle$

$\langle 1, 5 \rangle$ $\langle 10, -7 \rangle$

$\mathbb{R}^2$



Ex:

$$\mathbb{R}^5 = \{ \langle a_1, a_2, a_3, a_4, a_5 \rangle \mid a_1, a_2, a_3, a_4, a_5 \in \mathbb{R} \}$$

$$= \{ \langle 0,0,0,0,0 \rangle, \langle 1,1,1,1,1 \rangle, \langle 1,-1,5,3,2 \rangle, \langle 2.5, \pi, \tfrac{1}{2}, 0, 10 \rangle, \langle 2,3,4,5,13 \rangle, \ldots \}$$

infinitely many more

Ex: The length (or norm or magnitude) of a vector

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

is

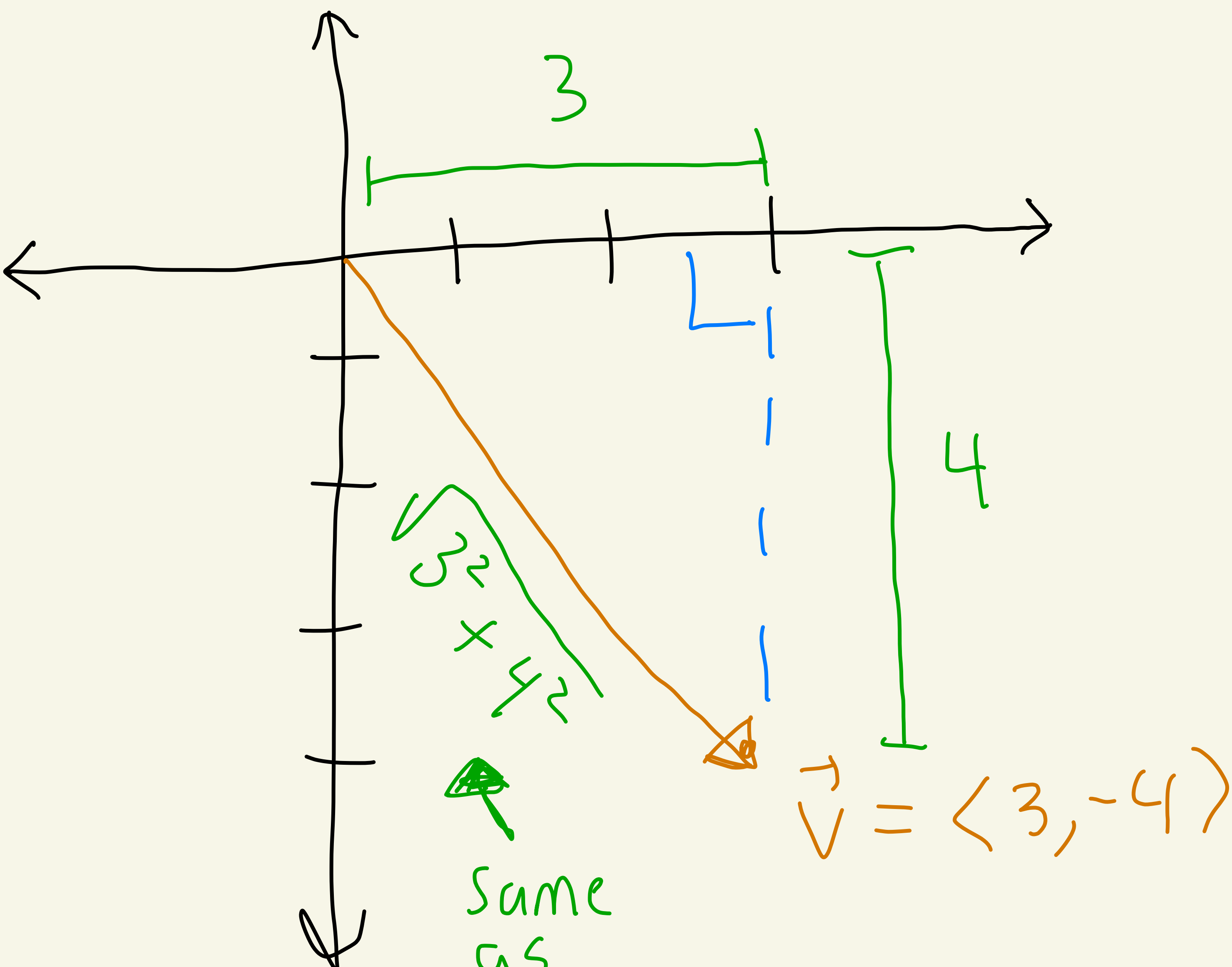
$$\underbrace{\|\vec{v}\|} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

some people use $|\vec{v}|$ instead of $\|\vec{v}\|$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 3, -4 \rangle$

Then, $\|\vec{v}\| = \sqrt{(3)^2 + (-4)^2}$

$= \sqrt{25} = 5$



Same as

$\sqrt{3^2 + (-4)^2}$

Ex: In $\mathbb{R}^6$ let

$$\vec{v} = \langle -1, 0, 2, 10, -3, 1 \rangle$$

Then

$$\|\vec{v}\| = \sqrt{(-1)^2 + (0)^2 + (2)^2 + (10)^2 + (-3)^2 + (1)^2}$$

$$= \sqrt{1 + 0 + 4 + 100 + 9 + 1}$$

$$= \sqrt{115}$$

$$\approx 10.7238\ldots$$

Operations on vectors

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$.
Let α be a $\underbrace{\text{scalar}}_{\text{means number}}$ in $\mathbb{R}$

some greek letters
α - alpha
β - beta
γ - gamma
δ - delta
ω - omega

Suppose

$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and

$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$.

Define

vector adding

$$\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle$$

$$\vec{v} - \vec{w} = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle$$

vector subtraction

$$\alpha \vec{v} = \langle \alpha a_1, \alpha a_2, \ldots, \alpha a_n \rangle$$

vector scaling

Ex: In $\mathbb{R}^3$

$$\langle 2,0,-1\rangle + \langle 1,\tfrac{1}{2},10\rangle$$
$$= \langle 2+1,\ 0+\tfrac{1}{2},\ -1+10\rangle$$
$$= \langle 3,\tfrac{1}{2},9\rangle$$

$$\langle 2,0,-1\rangle - \langle 1,\tfrac{1}{2},10\rangle$$
$$= \langle 2-1,\ 0-\tfrac{1}{2},\ -1-10\rangle$$
$$= \langle 1,-\tfrac{1}{2},-11\rangle$$

$$3\langle 2,0,-1\rangle$$
$$= \langle 3\cdot 2,\ 3\cdot 0,\ 3\cdot(-1)\rangle$$
$$= \langle 6,0,-3\rangle$$

Ex: In $\mathbb{R}^5$,

$$-2\langle 1, 0, \tfrac{1}{2}, 3, -1\rangle + \langle 2, 1, 3, -1, 5\rangle$$

$$= \langle -2, 0, -1, -6, 2\rangle + \langle 2, 1, 3, -1, 5\rangle$$

$$= \langle -2+2, 0+1, -1+3, -6-1, 2+5\rangle$$

$$= \langle 0, 1, 2, -7, 7\rangle$$

Def: The zero vector in $\mathbb{R}^n$, denoted by $\vec{0}$, is the vector containing all 0's.

Ex: In $\mathbb{R}^2$, $\vec{0} = \langle 0, 0\rangle$

In $\mathbb{R}^3$, $\vec{0} = \langle 0, 0, 0\rangle$

In $\mathbb{R}^4$, $\vec{0} = \langle 0, 0, 0, 0\rangle$

And so on...

Properties of vectors

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$.

Let α, β be $\underbrace{\text{scalars}}_{\text{numbers}}$ in $\mathbb{R}$.

Then:

① $\vec{u} + \vec{w} = \vec{w} + \vec{u}$ ← (cummutitive)

② $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ← (associative)

③ $\alpha(\beta \vec{u}) = (\alpha\beta)\vec{u}$

④ $(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$

⑤ $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

⑥ $\vec{u} + \vec{0} = \vec{u}$

$\vec{0} + \vec{u} = \vec{u}$

⑦ $\vec{u} + (-\vec{u}) = \vec{0}$

$-\vec{u} + \vec{u} = \vec{0}$

Ex:

$5(2\vec{u}) = (10)\vec{u}$

$5\vec{u} = 2\vec{u} + 3\vec{u}$

$2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v}$

Proof of 4 when $n=2$:

Let $\vec{u}$ be in $\mathbb{R}^2$

Let α, β be in $\mathbb{R}$.

Then, $\vec{u} = \langle a_1, a_2 \rangle$ where

a_1, a_2 are real numbers.

Then,

$$
\begin{aligned}
(\alpha+\beta)\vec{u} &= (\alpha+\beta)\langle a_1, a_2 \rangle \\
&= \langle (\alpha+\beta)a_1, (\alpha+\beta)a_2 \rangle \\
&= \langle \alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2 \rangle \\
&= \langle \alpha a_1, \alpha a_2 \rangle + \langle \beta a_1, \beta a_2 \rangle \\
&= \alpha \langle a_1, a_2 \rangle + \beta \langle a_1, a_2 \rangle \\
&= \alpha \vec{u} + \beta \vec{u} \qquad \square
\end{aligned}
$$