Math 2550-04

9/16/24

Fact: Applying an elementary row operation to a system of linear equations doesn't change the solution space to the system

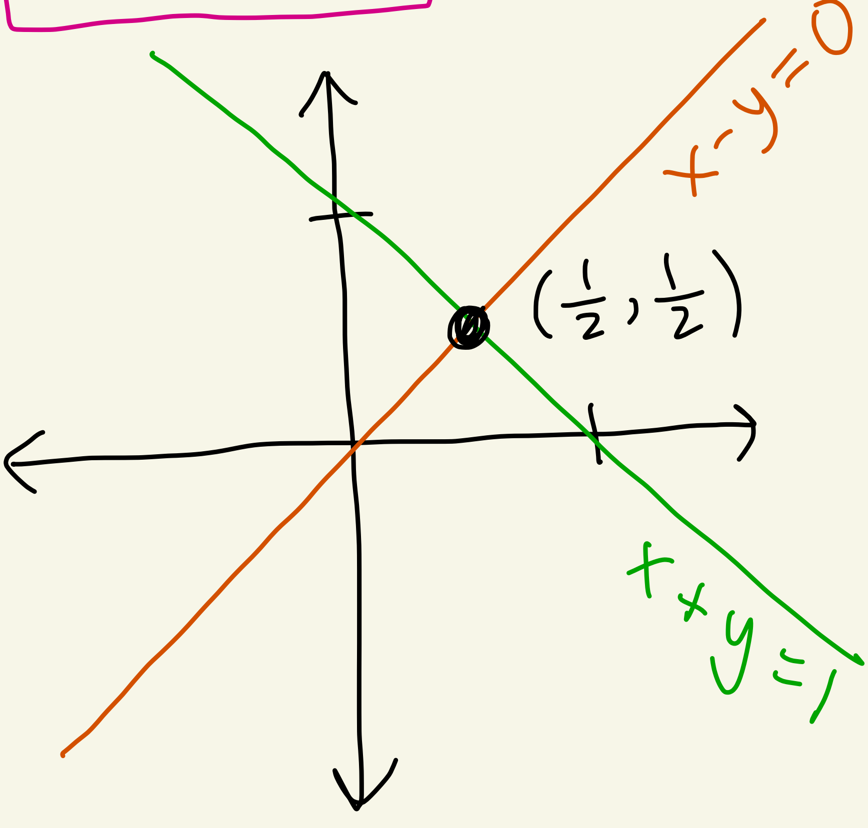
Ex:

system

$$\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}}$$

solution space

$$(x,y)=\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

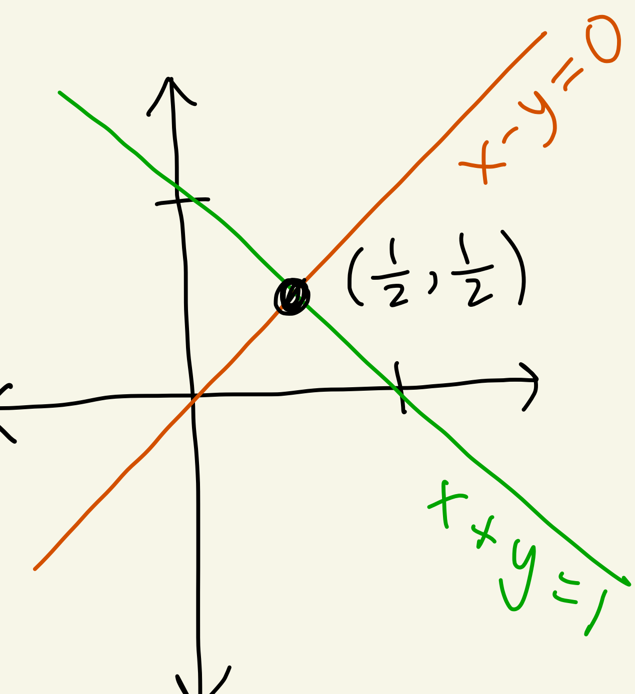


Let's apply an elementary row op.

$$\boxed{\begin{aligned} x+y&=1\\ x-y&=0 \end{aligned}} \xrightarrow{-R_1+R_2\to R_2} \boxed{\begin{aligned} x+y&=1\\ -2y&=-1 \end{aligned}}$$

$$\begin{array}{rl} -x-y=-1 & \leftarrow \boxed{-R_1} \\ +\ \ x-y=0 & \leftarrow \boxed{R_2} \\ \hline -2y=-1 & \leftarrow \boxed{\text{new } R_2} \end{array}$$

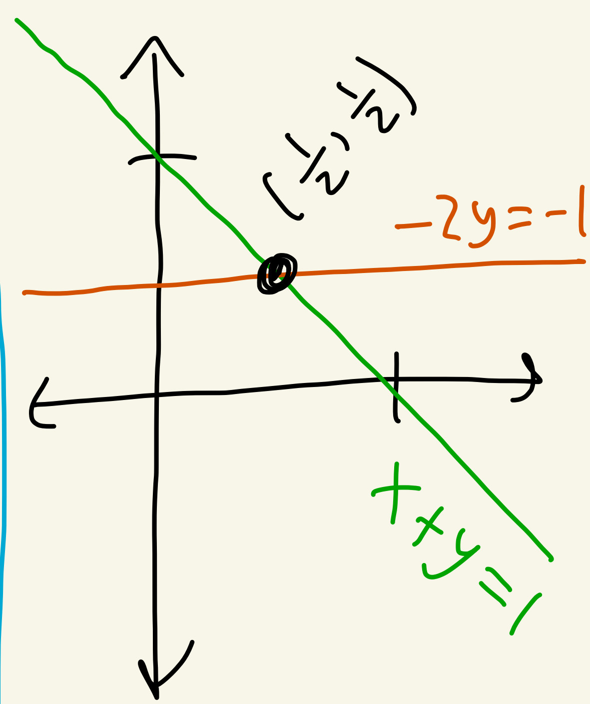
original



$$\boxed{\begin{aligned} x+y&=1\\ x-y&=0 \end{aligned}}$$

$$\xrightarrow{-R_1+R_2\to R_2}$$

new



$$\boxed{\begin{aligned} x+y&=1\\ -2y&=-1 \end{aligned}}$$

The equations are different but the solution space is the same.

<u>Def:</u> If a row of a matrix does not consist entirely of zeros then the <u>leading entry</u> in that row is the first non-zero entry when scanning from left to right.

<u>Ex:</u>

$$
A = \begin{pmatrix} 7 & 0 & -1 & 3 \\ 0 & 0 & 10 & 5 \\ 0 & -1 & 4 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \end{array}
$$

leading entry in row 1 is 7
leading entry in row 2 is 10
leading entry in row 3 is −1

There is no leading entry in row 4

Def: A matrix is in row echelon form if the following three conditions are true:

1. If there are any rows consisting of all zeros, then those rows are at the bottom of the matrix.
2. In any two consecutive rows that do not consist entirely of zeros, then the leading entry in the lower row occurs farther to the right than the leading entry in the upper row.
3. If a row doesn't consist entirely of zeros, then its leading entry is 1.

Ex:

$$\begin{pmatrix} 1 & 3 & -1 & 5 \\ 0 & 5 & 1 & 0 \end{pmatrix}$$

leading entries circled

1 ✓
2 ✓
3 ✗

not in row echelon form

Ex:

$$\begin{pmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

leading entries circled

1 ✓
2 ✓
3 ✓

In row echelon form

Ex:

$$\begin{pmatrix} 1 & 0 & 2 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1 ✓
2 ✗
3 ✓

not in row echelon form

leading entries are circled

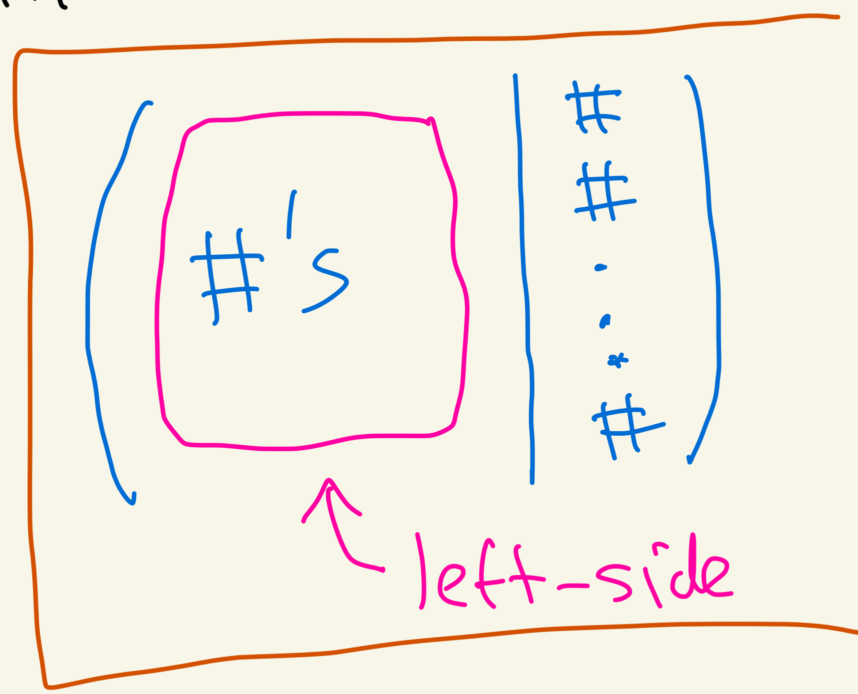
Ex: (flip R_2 and R_3)

$$\begin{pmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1 ✓
2 ✓
3 ✓

row echelon form

Def: Suppose you have an augmented matrix for a system of linear equations.
Suppose you use elementary row operations to put the left-side of the matrix into row echelon form.



The variable corresponding to the leading entry of a row is called a **leading variable** (or **pivot variable**)

Any variable that doesn't occur as a leading variable is called a **free variable**.

Ex:

Suppose

$$\underbrace{\begin{pmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{left-side is in row echelon form}} \left| \begin{matrix} 3 \\ 2 \end{matrix} \right)$$

leading entries circled

corresponds to the system

$$\boxed{\begin{aligned} x - y + 5z &= 3 \\ z &= 2 \end{aligned}}$$

leading variables: x, z

free variables: y

Ex: Suppose

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

leading entries circled

left-side is in row echelon form

corresponds to

$$\boxed{\begin{aligned} a + 2b + 5c &= 3 \\ b - c &= 2 \\ c &= 0 \end{aligned}}$$

leading variables: a, b, c

free variables: none

Method to solve a system of linear equations (called Gaussian elimination)

① Use elementary row operations to put the left side of the augmented matrix for the system into row echelon form.

② Look at the equations that go with this reduced system.

<u>case (a)</u>: If one of the equations is $0 = c$ where $c \neq 0$, then the system has no solutions.

<u>case (b)</u>: If case(a) doesn't occur then we use

"back substitution" to solve the system as follows:

(i) Solve each equation for it's leading variable.

(ii) Assign each free variable a new name.

(iii) Beginning with the bottom/last equation, successively substitute each equation into the equation above it.