Math 2550-04
9/23/24

Ex: Solve

$$\boxed{\begin{array}{rcl} a + 3b - 2c \quad\quad + 2e & = & 0 \\ 2a + 6b - 5c - 2d + 4e - 3f & = & -1 \\ 5c + 10d \quad\quad + 15f & = & 5 \\ 2a + 6b \quad\quad + 8d + 4e + 18f & = & 6 \end{array}}$$

We get that:

already 1 here

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

make these 0

$-2R_1 + R_2 \to R_2$

$-2R_1 + R_4 \to R_4$

$$\rightarrow \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make 1

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make these 0

$$\xrightarrow[-4R_2 + R_4 \to R_4]{-5R_2 + R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

make 1

$$\xrightarrow{\frac{1}{6}R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form

Back to equations:

$$\boxed{\begin{array}{rcl} a + 3b - 2c + 2e & = & 0 \\ c + 2d + 3f & = & 1 \\ f & = & \frac{1}{3} \\ 0 & = & 0 \end{array}}$$

leading variables

a, c, f

free variables

b, d, e

Solve for leading variables and give the free variables new names.

$$\boxed{\begin{aligned} a &= -3b+2c-2e \quad (1)\\ c &= 1-2d-3f \quad (2)\\ f &= 1/3 \quad (3)\\ b &= s \quad (4)\\ d &= t \quad (5)\\ e &= u \quad (6)\end{aligned}}$$

Back substitute:

(6) $e = u$

(5) $d = t$

(4) $b = s$

(3) $f = 1/3$

(2) $c = 1-2d-3f = 1-2t-3\left(\frac{1}{3}\right) = -2t$

1. $a = -3b + 2c - 2e = -3s + 2(-2t) - 2u$
$= -3s - 4t - 2u$

Answer:

$a = -3s - 4t - 2u$
$b = s$
$c = -2t$
$d = t$
$e = u$
$f = 1/3$

where s, t, u can be any real numbers

There are an infinite # of solutions

For example, if $s = 1$, $t = 0$, $u = 2$ we get this solution

$(a,b,c,d,e,f) = (-7, 1, 0, 0, 2, 1/3)$

Theorem: A system of linear equations has either

(i) no solutions,

(ii) exactly one solution,

or (iii) infinitely many solutions

(i) You get something like $0=5$

(iii) when you have free variables

Topic 4 – The inverse of a matrix

When you have a non-zero number like 4 you can find its inverse under multiplication. The inverse is $4^{-1} = \frac{1}{4}$. The property is

$$4 \cdot 4^{-1} = 4 \cdot \frac{1}{4} = 1$$

Our idea is to try to do this with square matrices and 1 is subbed with the identity matrix I_n.

Def: Let A be an $n \times n$ matrix [Ie A is a square matrix]

We say that A is invertible if there exists an $n \times n$ matrix B where

$$AB = I_n$$

and

$$BA = I_n$$

If $AB = I_n = BA$, then we say that A and B are inverses of each other.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

and $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.

I claim that A and B are inverses of each other.

Let's check it.

We get

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (1 \;\; 1) \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & (1 \;\; 1) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ (2 \;\; 1) \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} & (2 \;\; 1) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & 1-1 \\ -2+2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Also,

$$BA = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1+1 \\ 2-2 & 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Thus, $AB = I_2$

and $BA = I_2$.

So, A and B are both invertible and are inverses of each other.