Math 2550-04

9/25/24

<u>Theorem</u> (Inverses are unique)

Let A be an $n \times n$ invertible matrix, then there exists only one $n \times n$ matrix B where

$$AB = I_n = BA.$$

<u>Notation:</u> If A is invertible and $AB = I_n = BA$ then we write $B = A^{-1}$

<u>Ex:</u> We showed last time that $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

How to find A^{-1} if it exists

Let A be an $n \times n$ matrix.

Start with the matrix

$$\left(A \;\middle|\; I_n \right)$$

Apply elementary row operations until the left side of the above matrix is either I_n or has a row of zeros.

If you end up with I_n on the left side then A^{-1} will be on the right side.

If you get a row of zeros on the left side then A^{-1} does not exist.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Find A^{-1} (we already know it exists from Mon.)

We get:

want to try to make this I_2

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right)$$

$\underbrace{}_{A}$ $\underbrace{}_{I_2}$

make this 0

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

make this 0

$$\xrightarrow{-R_2 + R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \qquad \underbrace{\begin{matrix} -1 & 1 \\ 2 & -1 \end{matrix}}_{A^{-1}}$$

So, $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Ex: Let $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$

Find A^{-1} if it exists.

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want to make this I_2

$$\left(\begin{array}{cc|cc} \underbrace{\begin{matrix} 1 & 5 \\ -2 & -10 \end{matrix}}_{A} & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \end{array}\right)$$

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

row of zeros on left side

A does not have an inverse.

There is no matrix A^{-1} where $A^{-1}A = I_2$.

<u>Ex</u>: Find A^{-1} if it exists when $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$.

We get:

$$\left(\overbrace{\begin{matrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{matrix}}^{A} \;\middle|\; \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I_3}\right)$$

make this 1

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[-5R_2+R_3\to R_3]{-R_2+R_1\to R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & -1 & \frac{5}{3} & -3 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_3\to R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

make these 0

$$\xrightarrow[-R_3+R_2\to R_2]{-R_3+R_1\to R_1}\left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I_3}\middle|\underbrace{\begin{array}{ccc} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -\frac{5}{3} & 3 & -1 \end{array}}_{A^{-1}}\right)$$

Thus, A has an inverse and

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

<u>Theorem:</u> Let A and B be $n \times n$ matrices that are both invertible [ie A^{-1} and B^{-1} exist]. Then:

① AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$

② A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T$$

Sometimes we can solve a system of linear equations using an inverse matrix.
But first you have to write the system as a matrix equation.

Given the system

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad & \vdots \quad\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}} \quad (*)$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then (*) is equivalent to the matrix equation

$$\underbrace{A\vec{x}}_{\text{matrix multiplication}} = \vec{b}$$

Ex: Consider $\boxed{\begin{aligned} 2x - 3y &= 1 \\ x + 4y &= 7 \end{aligned}}$

Let

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Then,

$$A\vec{x} = \vec{b}$$

becomes

$$\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

which gives

$$\begin{pmatrix} 2x - 3y \\ x + 4y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

This is the same as

$$2x - 3y = 1$$
$$x + 4y = 7$$