2550

HW 4 — Part 1

Solutions

① (a)

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 - 1\cdot 0 & 1\cdot\left(\frac{1}{2}\right) - 1\cdot\left(\frac{1}{2}\right) \\ 0\cdot 1 + 2\cdot 0 & 0\cdot\left(\frac{1}{2}\right) + 2\cdot\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and

$$BA = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + \frac{1}{2}\cdot 0 & 1\cdot(-1) + \frac{1}{2}\cdot 2 \\ 0\cdot 1 + \frac{1}{2}\cdot 0 & 0(-1) + \frac{1}{2}\cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

since $AB = BA = I$

we have that A and B are inverses of each other.

① (b)

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 0+0\cdot 1+1\cdot 1 & 1\cdot 1+0\cdot 0+1\cdot 2 & 1\cdot(-1)+0\cdot 1+1\cdot 1 \\ 1\cdot 0+1\cdot 1+2\cdot 1 & 1\cdot 1+1\cdot 0+2\cdot 2 & 1\cdot(-1)+1\cdot 1+2\cdot 1 \\ 0\cdot 0+1\cdot 1+1\cdot 1 & 0\cdot 1+1\cdot 0+1\cdot 2 & 0(-1)+1\cdot 1+1\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 0 \\ 3 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \neq I$$

Since $AB \neq I$,

A and B are not inverses of each other.

2 (a)

already have a 1 here

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

make this a zero

make this a 1

make this a zero

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{-4R_2+R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

So, $\begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$

(2) (b) Find A^{-1} where

$$A = \begin{pmatrix} -3 & 6 \\ 4 & 5 \end{pmatrix}$$

make this a 1

$$\left(\begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & -2 & -1/3 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right)$$

make this a zero

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & -2 & -1/3 & 0 \\ 0 & 13 & 4/3 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{13}R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & -2 & -1/3 & 0 \\ 0 & 1 & 4/39 & 1/13 \end{array}\right)$$

make this a zero

$$\xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{-5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array}\right)$$

So,

$$A^{-1} = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

2 (c)

make this a 1

$$\left(\begin{array}{cc|cc} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{6}R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & -\frac{2}{3} & 1/6 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right)$$

make this a zero

$$\xrightarrow{3R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & -\frac{2}{3} & 1/6 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right)$$

we got a row of zeros

No inverse exists because we got a row of zeros.

The inverse of $\begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$ does not exist.

3 (a)

make this a 1

$$\left(\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{-5R_2+R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 1/4 & -3/4 & 0 \\ 0 & 0 & \frac{5}{2} & -\frac{5}{4} & \frac{7}{4} & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{2}{5}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 1/4 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right)$$

make these zeros

$$\xrightarrow[\frac{10}{4}R_3 + R_2 \to R_2]{-3R_3 + R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right)$$

So,

$$\begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{pmatrix}$$

③ (b)

make this a 1

$$\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow[4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[10R_2+R_3 \to R_3]{3R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{19}{10} & -\frac{2}{5} & \frac{3}{10} & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

row of zeros
no inverse exists

Since we got a row of zeros on the left side, the inverse of $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ does not exist.

③ (c)

already have a 1 here

make these zeros

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_1+R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

already have a 1 here

make these zeros

$$\xrightarrow{-R_2+R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-\frac{1}{2}R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

make these zeros

$$\xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

Thus,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

3 (d)

make this a 1

$$\left(\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right)$$

already have a 1 here

$$\xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow[-R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

already have a 1

make these zeros

$$\xrightarrow{-3R_3+R_1\to R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

Thus,

$$\begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

③ (e)

already have a 1 here

make these zeros

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

already have a 1 here

make these zeros

$$\xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

make these zeros

$$\xrightarrow[\substack{-2R_3+R_2\to R_2}]{-R_3+R_1\to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

Thus,

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

4 (a)

already have a 1 here

make these into zeros

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow[-R_1+R_4 \to R_4]{\substack{-R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-3R_2+R_4 \to R_4]{-3R_2+R_3 \to R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{5}R_3 \to R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{-5R_3 + R_4 \to R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{7}R_4 \to R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{array}\right)$$

So,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{1}{7} \end{pmatrix}$$

④ (b)

$$\begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix} \leftarrow$$

already has a row of zeros.

No inverse exists since it has a row of zeros.

5 (a)

$$\boxed{\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}}$$

corresponds to

$$\boxed{\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}} \quad (*)$$

Let's find the inverse of $\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}$ if it exists.

already have a 1 here

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \xrightarrow{-5R_1+R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right)$$

make this a zero

already have a 1 here

make this a zero

$$\xrightarrow{-R_2+R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right)$$

Thus, $\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & -1 \\ -5 & 1 \end{pmatrix}$

Apply the inverse to the equation (*) on the previous page.

$$\underbrace{\begin{pmatrix} 6 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}}_{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

So, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (6)(2) + (-1)(9) \\ (-5)(2) + (1)(9) \end{pmatrix}$

Thus,

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$

5 (b)

$$\boxed{\begin{aligned} 4x_1 - 3x_2 &= -3 \\ 2x_1 - 5x_2 &= 9 \end{aligned}}$$

corresponds to

$$\boxed{\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}} \quad (*)$$

Let's find the inverse of $\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}$ if it exists.

make this a 1

$$\left(\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 2 & -5 & 0 & 1 \end{array}\right)$$

make this a zero

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{7}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-\frac{2}{7}R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right)$$

make this a zero

$$\xrightarrow{\frac{3}{4}R_2 + R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{5}{14} & -\frac{3}{14} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right)$$

So,
$$\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}.$$

Now apply the inverse to both sides of $(*)$

$$\underbrace{\begin{pmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}}_{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

So,
$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \left(\frac{5}{14}\right)(-3) + \left(-\frac{3}{14}\right)(9) \\ \left(\frac{1}{7}\right)(-3) + \left(-\frac{2}{7}\right)(9) \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}}$$

5 (c)

$$\boxed{\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}}$$

Corresponds to

$$\boxed{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}} \quad (*)$$

Let's find the inverse of $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ if it exists.

we have a 1 here

Make these zeros

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow[-2R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow[3R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-4R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right)$$

make these zeros

$$\xrightarrow[-\frac{1}{4}R_3+R_2 \to R_2]{-\frac{1}{4}R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right)$$

So, $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}$

Apply the inverse to both sides of (*) to get:

$$\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}}_{I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

So,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} (-1)(4)+(0)(-1)+(1)(3) \\ (0)(4)+(-1)(-1)+(1)(3) \\ (2)(4)+(3)(-1)+(-4)(3) \end{pmatrix}$$

So, $\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}}$

5 (d)

$$\boxed{\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}}$$

corresponds to

$$\boxed{\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}} \quad (*)$$

Let's find the inverse of $\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ if it exists.

put a 1 here

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{6}{5} & \frac{4}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{5}{6}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

make these zero

$$\xrightarrow[-R_2 + R_3 \to R_3]{-\frac{3}{5}R_2 + R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{3R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right)$$

make these zeros

$$\xrightarrow{-\frac{2}{3}R_3 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right)$$

So, $\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}$

Apply the inverse to both sides of $(*)$ to get:

$$\underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}}_{I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

So,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{4}{2} - 1 + 0 \\ -\frac{12}{2} + 5 - 10 \\ \frac{12}{2} - 5 + 15 \end{pmatrix}$$

So,

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \\ 16 \end{pmatrix}}$$