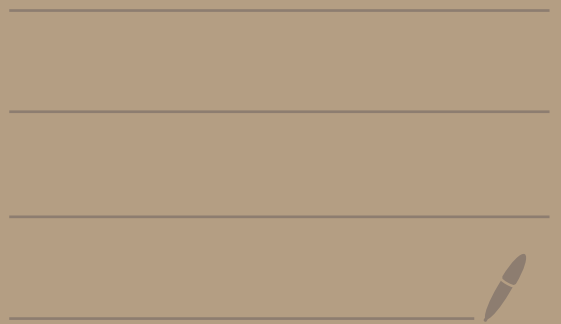


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HW 5 - Part 2

Solutions

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1(a) This isn't always true.

Consider  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  and  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then

$$\det(I+A) = \det \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 9$$

but

$$1 + \det(A) = 1 + \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 1 + 4 = 5$$

And  $9 \neq 5$ .

So in this example,

$$\det(I+A) \neq 1 + \det(A)$$

1(b)

The statement

$$\det(AB - BA) = 0$$

is not always true.

For example, set

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

$$\text{Then } AB = \begin{pmatrix} 5 & 8 \\ 7 & 10 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 11 & 10 \\ 5 & 4 \end{pmatrix}.$$

$$\text{So, } AB - BA = \begin{pmatrix} -6 & -2 \\ 2 & 6 \end{pmatrix}.$$

And

$$\begin{aligned} \det(AB - BA) &= \det \begin{pmatrix} -6 & -2 \\ 2 & 6 \end{pmatrix} \\ &= (-6)(6) - (-2)(2) \\ &= -32 \neq 0 \end{aligned}$$

1(c)

The statement is not always true.

For example, set  $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ .

Then,

$$\det(4A) = \det \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} = 12^2 = 144$$

but

$$4 \cdot \det(A) = 4 \cdot \det \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 4 \cdot 3^2 = 36$$

and  $144 \neq 36$ .

So for this example,

$$\det(4A) \neq 4 \det(A)$$

1(d) The statement

$$\det(A+B) = \det(A) + \det(B)$$

is not always true.

For example, set

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A+B) &= \det \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} = (2)(2) - (3)(4) \\ &= 4 - 12 = -8 \end{aligned}$$

but

$$\begin{aligned} \det(A) + \det(B) &= \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \\ &= \left[ (1)(1) - (1)(1) \right] + \left[ (1)(1) - (2)(3) \right] = 0 - 6 = -6 \end{aligned}$$

So, in this case  $\det(A+B) \neq \det(A) + \det(B)$ .

$2(a)$

Let  $\alpha$  be a real number and  
 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a  $2 \times 2$  matrix.

Then,

$$\begin{aligned} \det(\alpha A) &= \det\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \det\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \\ &= (\alpha a)(\alpha d) - (\alpha b)(\alpha c) \\ &= \alpha^2 ad - \alpha^2 bc \\ &= \alpha^2 [ad - bc] \\ &= \alpha^2 \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \alpha^2 \det(A) \end{aligned}$$

② (b) Let  $A$  and  $B$  be  $2 \times 2$  matrices.  
So,  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ .

Then,

$$\det(AB) = \det \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$= (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$

$$= \cancel{aef} + aedh + bgcf + \cancel{bgdh} \\ - \cancel{afce} - afdg - bhce - \cancel{bhdg}$$

$$= aedh + bgcf - afdg - bhce$$

And,

$$\det(A)\det(B) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= (ad - bc)(eh - fg) = aedh - afdg - bceh + bcfg \\ = aedh + bgcf - afdg - bhce$$

Comparing the above two expressions we

see that  $\det(AB) = \det(A)\det(B)$ .

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$$\det \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= a \cdot \begin{vmatrix} 0 & d \\ 0 & e \end{vmatrix} - 0 + 0$$

$$\begin{array}{|c|c|c|} \hline \cancel{a} & b & c \\ \hline 0 & 0 & d \\ \hline 0 & 0 & e \\ \hline \end{array}$$

$$= a [(0)(e) - (d)(0)]$$

$$= a [0] = 0$$

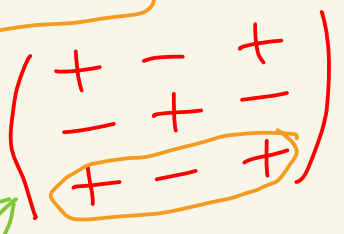


4 To find when A is not invertible  
 We need to find all k such that  
 $\det(A) = 0$ .

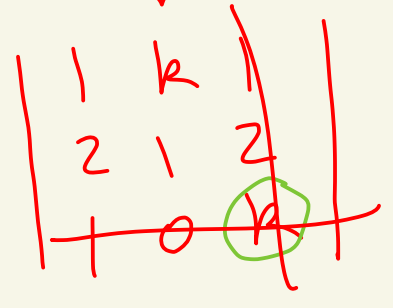
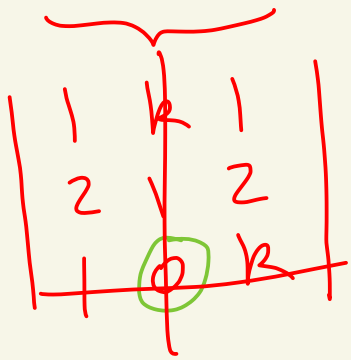
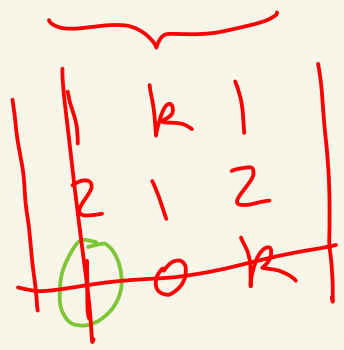
We have that

$$\det(A) = \begin{vmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{vmatrix}$$

Expand on row 3



$$= 1 \cdot \begin{vmatrix} k & 1 \\ 1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + k \cdot \begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix}$$



$$= 1 \cdot [(k)(2) - (1)(1)] - 0 + k [(1)(1) - (k)(2)]$$

$$= 2k - 1 + k - 2k^2 = -2k^2 + 3k - 1$$

When is  $-2k^2 + 3k - 1 = 0$  ?

$$\text{When } k = \frac{-3 \pm \sqrt{3^2 - 4(-2)(-1)}}{2(-2)} = \frac{-3 \pm \sqrt{9-8}}{-4}$$
$$= \frac{-3 \pm 1}{-4}$$

$$\text{When } k = \frac{-3+1}{-4} = \frac{-2}{-4} = \frac{1}{2} \quad \text{OR} \quad k = \frac{-3-1}{-4} = \frac{-4}{-4} = 1$$

$$\text{So, } A = \begin{pmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{pmatrix}$$

is not invertible when  
 $\det(A) = 0$  which is  
when  $k = \frac{1}{2}$  or  $k = 1$ .