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HW 5 - Part 2

Solutions

$\boxed{1(a)}$ This isn't always true.

Consider $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then

$$\det(I+A) = \det\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 9$$

but

$$1 + \det(A) = 1 + \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 1+4 = 5$$

And $9 \neq 5$.

So in this example,

$$\det(I+A) \neq 1 + \det(A)$$

$\boxed{1(b)}$

The statement

$$\det(AB-BA) = 0$$

is not always true.

For example, set

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 5 & 8 \\ 7 & 10 \end{pmatrix}$ and $BA = \begin{pmatrix} 11 & 10 \\ 5 & 4 \end{pmatrix}$.

So, $AB - BA = \begin{pmatrix} -6 & -2 \\ 2 & 6 \end{pmatrix}$.

And

$$\det(AB-BA) = \det\begin{pmatrix} -6 & -2 \\ 2 & 6 \end{pmatrix}$$
$$= (-6)(6) - (-2)(2)$$
$$= -32 \neq 0$$

$\boxed{1(c)}$

The statement is not always true.
For example, set $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Then,

$$\det(4A) = \det\begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} = 12^2 = 144$$

but

$$4 \cdot \det(A) = 4 \cdot \det\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 4 \cdot 3^2 = 36$$

and $144 \neq 36$.

So for this example,

$$\det(4A) \neq 4 \det(A)$$

$\boxed{1(d)}$ The statement

$$\det(A+B) = \det(A) + \det(B)$$

is not always true.

For example, set

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Then

$$\det(A+B) = \det\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} = (2)(2) - (3)(4)$$
$$= 4 - 12 = -8$$

but

$$\det(A) + \det(B) = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$
$$= \left[(1)(1) - (1)(1)\right] + \left[(1)(1) - (2)(3)\right] = 0 - 6 = -6$$

So, in this case

$$\det(A+B) \neq \det(A) + \det(B).$$

$\boxed{2(a)}$

Let α be a real number and

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.

Then,

$$\det(\alpha A) = \det\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \det\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

$$= (\alpha a)(\alpha d) - (\alpha b)(\alpha c)$$

$$= \alpha^2 ad - \alpha^2 bc$$

$$= \alpha^2 [ad - bc]$$

$$= \alpha^2 \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \alpha^2 \det(A)$$

(2) (b) Let A and B be 2×2 matrices.

So, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Then,

$$\det(AB) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$= (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$

$$= \cancel{aecf} + aedh + bgcf + \cancel{bgdh} - \cancel{afce} - afdg - bhce - \cancel{bhdg}$$

$$= aedh + bgcf - afdg - bhce$$

And,

$$\det(A)\det(B) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\det\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= (ad-bc)(eh-fg) = adeh - adfg - bceh + bcfg$$

$$= aedh + bgcf - afdg - bhce$$

Comparing the above two expressions we see that $\det(AB) = \det(A)\det(B)$.

$\boxed{3}$

$$\det\begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= a \cdot \begin{vmatrix} 0 & d \\ 0 & e \end{vmatrix} - 0 + 0$$

$$\begin{vmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{vmatrix}$$

$$= a\left[(0)(e) - (d)(0)\right]$$

$$= a[0] = 0$$

$\boxed{4}$ To find when A is not invertible we need to find all k such that $\det(A) = 0$.

We have that

Expand on row 3

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{vmatrix}$$

$$= 1 \cdot \underbrace{\begin{vmatrix} k & 1 \\ 1 & 2 \end{vmatrix}}_{\begin{vmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{vmatrix}} - 0 \cdot \underbrace{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}_{\begin{vmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{vmatrix}} + k \cdot \underbrace{\begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix}}_{\begin{vmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{vmatrix}}$$

$$= 1 \cdot \left[(k)(2) - (1)(1)\right] - 0 + k\left[(1)(1) - (k)(2)\right]$$

$$= 2k - 1 + k - 2k^2 = -2k^2 + 3k - 1$$

When is $-2k^2+3k-1=0$?

When $k = \dfrac{-3 \pm \sqrt{3^2-4(-2)(-1)}}{2(-2)} = \dfrac{-3 \pm \sqrt{9-8}}{-4}$

$= \dfrac{-3 \pm 1}{-4}$

When $k = \dfrac{-3+1}{-4} = \dfrac{-2}{-4} = \dfrac{1}{2}$ or $k = \dfrac{-3-1}{-4} = \dfrac{-4}{-4} = 1$

So, $A = \begin{pmatrix} 1 & k & 1 \\ 2 & 1 & 2 \\ 1 & 0 & k \end{pmatrix}$

is not invertible when det$(A) = 0$ which is when $k = \frac{1}{2}$ or $k = 1$.