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HW 6 Part 2 Solutions

For all the problems we use this theorem from class:

Let V be a vector space over a field F and let W be a subset of V.
W is a subspace of V if and only if the following three conditions hold:

① $\vec{0}$ is in W — W contains $\vec{0}$

② If $\vec{v}$ and $\vec{w}$ are in W, then $\vec{v}+\vec{w}$ is in W. — W is closed under vector addition

③ If $\vec{z}$ is in W and α is in F, then $\alpha \cdot \vec{z}$ is in W. — W is closed under scalar multiplication

$\boxed{1(a)}$ $W = \{\langle a, 0, 0\rangle \mid a \in \mathbb{R}$

(i) $\langle 1, 0, 0\rangle$, $\langle \pi, 0, 0\rangle$
$\langle -\frac{1}{2}, 0, 0\rangle$, $\langle 321, 0, 0\rangle$

(ii) W is a subspace. Let's prove the three conditions.

① Setting $a=0$ gives $\langle 0,0,0\rangle$ is in W.
So, W contains the ~~zero~~ zero vector.

② Let $\vec{v}, \vec{w}$ be in W. Then
$\vec{v} = \langle a_1, 0, 0\rangle$ and $\vec{w} = \langle a_2, 0, 0\rangle$
where a_1 and a_2 are real numbers.
We have $\vec{v} + \vec{w} = \langle a_1 + a_2, 0, 0\rangle$
which is of the form of the elements in W. So $\vec{v}+\vec{w}$ is in W.
So, W is closed under vector addition.

(3) Let $\vec{z}$ be in W and α be in $F = \mathbb{R}$.
Then $\vec{z} = \langle a, 0, 0 \rangle$ for some real number a.
So, $\alpha \vec{z} = \langle \alpha a, 0, 0 \rangle$ is in W since its of the form of the elements in W.

So, W is closed under scalar multiplication.

Since W satisfies properties (1), (2), and (3) W is a subspace of $\mathbb{R}^3$.

$\boxed{1(b)}$ $W = \{\langle a, 1, 2\rangle \mid a \text{ in } \mathbb{R}\}$

(i) $\langle 1, 1, 2\rangle$, $\langle -3, 1, 2\rangle$

$\langle \pi, 1, 2\rangle$, $\langle -\sqrt{2}, 1, 2\rangle$

(ii) W is not a subspace. For example condition ① is not met since $\langle 0,0,0\rangle$ is not of the form $\langle a, 1, 2\rangle$.

So, W does not contain the zero vector.

The above is enough to be done with the problem.

Note: You could have also shown that conditions ② or ③ are not true.

For example, $\vec{v} = \langle 0, 1, 2\rangle$ and $\vec{w} = \langle 1, 1, 2\rangle$ are in W, but $\vec{v} + \vec{w} = \langle 1, 2, 4\rangle$ is not in W since its not of the form $\langle a, 1, 2\rangle$. So condition ② is not met.

That is W is not closed under vector addition.

Also, $\vec{z} = \langle 0, 1, 2\rangle$ is in W. But, $3\vec{z} = \langle 0, 3, 6\rangle$ is not in W. So, condition ③ is not met. That is, W is not closed under scalar multiplication.

$\boxed{1(c)}$ $W = \{\langle a,b,c\rangle \mid b = a+c,\ a,b,c \in \mathbb{R}\}$

(i) $\langle 1,3,2\rangle \in W$ since $3 = 1+2$

$\langle \pi, \pi-2, -2\rangle \in W$ since $\pi - 2 = \pi + (-2)$

$\langle 5, \frac{11}{2}, \frac{1}{2}\rangle \in W$ since $\frac{11}{2} = 5 + \frac{1}{2}$

$\langle 3, 0, -3\rangle \in W$ since $0 = 3 + (-3)$

(ii) W is a subspace of $\mathbb{R}^3$. Let's verify.

① $\langle a,b,c\rangle = \langle 0,0,0\rangle$ is in W since $b = a+c$ when $a=0, b=0, c=0$. So, W contains the zero vector.

Then

② Let $\vec{v}$ and $\vec{w}$ be in W.

$\vec{v} = \langle a_1, b_1, c_1\rangle$ and $\vec{w} = \langle a_2, b_2, c_2\rangle$

where $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$.

Adding these two equations gives

$$(b_1 + b_2) = (a_1 + a_2) + (c_1 + c_2)$$

Therefore,

$$\vec{v} + \vec{w} = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2\rangle$$

is in W. → Since its 2nd component equals the sum of the 1st and 3rd components

So, W is closed under vector addition

3 Let α be in $F = \mathbb{R}$ and $\vec{z}$ be in W.
Then $\vec{z} = \langle a, b, c \rangle$ where $b = a + c$.

So, $\alpha b = \alpha a + \alpha c$.

Thus,

$$\alpha \vec{z} = \langle \alpha a, \alpha b, \alpha c \rangle$$

is in W. [since its 2nd component equals the sum of the 1st and 3rd components]

So, W is closed under scalar multiplication.

$\boxed{1(d)}$ $W = \{\langle a,b,c\rangle \mid b = a+c+1,\ a,b,c \in \mathbb{R}\}$

(i) $\langle 1, 5, 3\rangle \in W$ since $5 = 1+3+1$

$\langle 0, 1, 0\rangle \in W$ since $1 = 0+0+1$

$\langle \pi, \pi+3, 2\rangle \in W$ since $\pi+3 = \pi+2+1$

$\langle \frac{1}{2}, 1, -\frac{1}{2}\rangle \in W$ since $1 = \frac{1}{2} - \frac{1}{2} + 1$

(ii) W is not a subspace.

1 For example, $\langle a,b,c\rangle = \langle 0,0,0\rangle$ is not in W since $b \neq a+c+1$. So condition 1 is in this example. not satisfied.

Therefore, W is not a subspace.

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The previous page shows that W does not contain the zero vector so W is not a subspace. This is enough to be done with the problem.

For illustrative purposes let me show you how you could have shown that W does not satisfy property 2 or 3

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2 Let $\vec{v} = \langle 1, 5, 3 \rangle$ and $\vec{w} = \langle 0, 1, 0 \rangle$

Then $\vec{v}$ and $\vec{w}$ are in W, since $5 = 1+3+1$ and $1 = 0+0+1$. However $\vec{v}+\vec{w} = \langle 1, 6, 3 \rangle$ is not in W since $6 \neq 1+3+1$. So, W is not closed under vector addition.

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3 Let $\vec{z} = \langle 1, 5, 3 \rangle$ and $\alpha = 2$.

Then $\vec{z}$ is in W since $5 = 1+3+1$. However, $\alpha\vec{z} = 2\vec{z} = \langle 2, 10, 6 \rangle$ is not in W since $10 \neq 2+6+1$. So, W is not closed under scalar multiplication.

$\boxed{\boxed{2(a)}}$ $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a+b+c+d=0 \right\}$

(i) $\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \in W$ since $1+(-2)+1+0=0$

$\begin{pmatrix} \pi & -2 \\ 2 & -\pi \end{pmatrix} \in W$ since $\pi+(-2)+2+\pi=0$

$\begin{pmatrix} 3 & \frac{1}{2} \\ \frac{1}{2} & -4 \end{pmatrix} \in W$ since $3+\frac{1}{2}+\frac{1}{2}-4=0$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in W$ since $1+(-1)+(-1)+1=0$

(ii) We show that W is a subspace of $V = M_{2,2}$.

① Setting $a=0, b=0, c=0, d=0$ gives $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and here we have that $a+b+c+d = 0+0+0+0=0$.

Thus, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W.

② Let $\vec{v}, \vec{w}$ be in W.

Then $\vec{v} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ where $a_1 + b_1 + c_1 + d_1 = 0$

and $\vec{w} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ where $a_2 + b_2 + c_2 + d_2 = 0$.

Adding the two equations yields

$$(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) + (d_1 + d_2) = 0$$

Thus,

$\vec{v} + \vec{w} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$ is in W.

So, W is closed under vector addition.

[Because the sum of its four entries is 0, which is what characterizes the elements of W]

③ Let α be a scalar in $F = \mathbb{R}$.

Let $\vec{z} \in W$.

Then $\vec{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a+b+c+d=0$.

Multiplying the above equation by α yields

$$(\alpha a) + (\alpha b) + (\alpha c) + (\alpha d) = 0.$$

Hence,

$$\alpha \vec{z} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$ is in W.

So, W is closed under scalar multiplication.

Because the sum of its four entries is 0, which is what characterizes the elements of W

$\boxed{\boxed{2(b)}}$ $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$

(i) $\det \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} = (1)(-2) - (2)(-1) = 0$

So, $\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \in W.$

$\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (1)(1) - (1)(1) = 0$

So, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in W.$

$\det \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = (1)(\tfrac{1}{2}) - (\tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) = 0$

So, $\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \in W.$

$\det \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} = (0)(5) - (0)(0) = 0$

So, $\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \in W.$

(ii) W is not a subspace, however it does satisfy the first condition.

① $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ since $\det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (0)(0) - (0)(0) = 0$

[So condition ① is satisfied]

② This condition is not satisfied.

For example, let $\vec{v} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$\det(\vec{v}) = (1)(0) - (-1)(0) = 0$

and

$\det(\vec{w}) = (0)(1) - (0)(1) = 0$

So, $\vec{v}$ and $\vec{w}$ are in W.

However, $\det(\vec{v}+\vec{w}) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = (1)-(-1) = 2 \neq 0$

So, $\vec{v}+\vec{w}$ is not in W.

Thus, W is not closed under addition and is not a subspace of $M_{2,2}$.

So, we saw that condition 1 is satisfied but condition 2 is not.
At this point you can stop and say that W is not a subspace of $M_{2,2}$ since condition 2 is not satisfied.

For illustrative purposes, let me show you how to show that condition 3 is also satisfied.

Let α be a scalar in $F = \mathbb{R}$ and let $\vec{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in W. Then,

$\det(\vec{z}) = ad - bc = 0$ since z is in W.

So, $\det(\alpha \vec{z}) = \det \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} =$

$= (\alpha a)(\alpha d) - (\alpha b)(\alpha c) = \alpha [ad - bc]$

$= \alpha(0) = 0$. Thus, $\alpha \vec{z}$ is in W.

$\boxed{\boxed{3a}}$ $W = \{ 1 + bx + cx^2 \mid b, c \text{ are in } \mathbb{R} \}$

(i)
$$\left.\begin{array}{l} 1 + 2x - 3x^2 \\ 1 + 0x + 0x^2 = 1 \\ 1 - x + \pi x^2 \\ 1 + 0 \cdot x + 1 \cdot x^2 = 1 + x^2 \end{array}\right\} \text{some elements of } W$$

(ii) W is not a subspace of $M_{2,2}$.

(1) $\vec{0} = 0 + 0x + 0x^2$ is not in W because it isn't of the form $1 + bx + cx^2$.

Therefore, W is not a subspace.

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We showed W is not a subspace on the previous page by showing property (1) is not satisfied. That is enough to be done with the problem.

For illustrative purposes, let me show you how to show that properties (2) and (3) are also not satisfied.

(2) Let $\vec{v} = 1 + x + x^2$ and $\vec{w} = 1 + 2x - x^2$ be in W. Then $\vec{v} + \vec{w} = 2 + 3x + 0x^2$ which is not of the form $1 + bx + cx^2$ so not in W. Thus, W is not closed under addition.

(3) Let $\vec{z} = 1 + x + x^2$ and $\alpha = 2$. Then $\vec{z}$ is in W. But $2\vec{z} = 2 + 2x + 2x^2$ which is not of the form $1 + bx + cx^2$ so in not in W. So, W is not closed under scalar multiplication.

$\boxed{\boxed{3b}}$ $W = \left\{ a + bx + cx^2 \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{R} \\ a + 2b = 0 \end{array} \right\}$

(i) $1 - \frac{1}{2}x + 7x^2$ is in W

because $1 + 2\left(-\frac{1}{2}\right) = 0$.

$3 - \frac{3}{2}x - \pi x^2$ is in W

because $3 + 2\left(-\frac{3}{2}\right) = 0$

$0 + 0x + 3x^2$ is in W

because $0 + 2(0) = 0$

$-\frac{1}{2} + \frac{1}{4}x - 10x^2$ is in W

because $-\frac{1}{2} + 2\left(\frac{1}{4}\right) = 0$

(ii) W is a subspace of P_2.

① Setting $a=0, b=0, c=0$ we get $a+bx+cx^2 = 0+0x+0x^2$ and $a+2b=0$.

Thus, $\vec{0} = 0+0x+0x^2$ is in W.

② Let $\vec{v}$ and $\vec{w}$ be in P_2.

Then, $\vec{v} = a_1+b_1x+c_1x^2$ where $a_1+2b_1=0$

and $\vec{w} = a_2+b_2x+c_2x^2$ where $a_2+2b_2=0$.

Adding these equations gives

$$(a_1+a_2)+2(b_1+b_2)=0.$$

Hence, $\vec{v}+\vec{w} = (a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2$ is in W.

So, W is closed under vector addition.

③ Let $\vec{z}$ be in W and α be in $F = \mathbb{R}$.

Then $\vec{z} = a + bx + cx^2$
where $a + 2b = 0$.

since $\vec{z}$ is in W

Multiplying $a + 2b = 0$ by α gives

$$(\alpha a) + 2(\alpha b) = 0.$$

Thus,

$$\alpha \vec{z} = \alpha a + \alpha b x + \alpha c x^2$$

is in W.

So, W is closed under scalar multiplication.