Math 2550-01

10/21/24

Ex: In $\mathbb{R}^3$, let

$\vec{a} = \langle -1, 0, 1 \rangle$, $\vec{b} = \langle -5, -3, 2 \rangle$, $\vec{c} = \langle 1, 1, 0 \rangle$

Then,

$$\underbrace{\langle -5, -3, 2 \rangle = 2\langle -1, 0, 1 \rangle - 3\langle 1, 1, 0 \rangle}_{\vec{b} = 2\vec{a} - 3\vec{c}}$$

Since $\vec{b}$ is a linear combination of $\vec{a}$ and $\vec{c}$ we say that $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

Note we can rewrite the above

$$2\vec{a} - 1 \cdot \vec{b} - 3\vec{c} = \vec{0}$$

Theorem: The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are linearly independent if and only if the only solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0}$$

is $c_1 = 0, c_2 = 0, \ldots, c_r = 0$.

If there are more solutions then the vectors are linearly dependent.

Ex: Are $\vec{v}_1 = \langle 1, -2, 1 \rangle$, $\vec{v}_2 = \langle 1, 0, 1 \rangle$, $\vec{v}_3 = \langle 0, 1, 0 \rangle$ linearly dependent or linearly independent?

We need to solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

that is

$$c_1\langle 1,-2,1\rangle + c_2\langle 1,0,1\rangle + c_3\langle 0,1,0\rangle = \langle 0,0,0\rangle$$

We get

$$\langle c_1,-2c_1,c_1\rangle + \langle c_2,0,c_2\rangle + \langle 0,c_3,0\rangle = \langle 0,0,0\rangle$$

This gives

$$\langle c_1+c_2,\ -2c_1+c_3,\ c_1+c_2\rangle = \langle 0,0,0\rangle$$

We get

$$\boxed{\begin{aligned} c_1 + c_2 \quad &= 0 \\ -2c_1 \quad + c_3 &= 0 \\ c_1 + c_2 \quad &= 0 \end{aligned}}$$

Solving we get

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{1/2\, R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 \quad &= 0 \quad (1) \\ c_2 + \tfrac{1}{2} c_3 &= 0 \quad (2) \\ 0 &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{leading: } c_1, c_2 \\ &\text{free: } c_3 \end{aligned}}$$

So,

$$\boxed{\begin{aligned} c_1 &= -c_2 \quad (1) \\ c_2 &= -\tfrac{1}{2} c_3 \quad (2) \end{aligned}}$$

$$\boxed{c_3 = t} \quad ③$$

Back substitution gives

③ $c_3 = t$

② $c_2 = -\frac{1}{2}c_3 = -\frac{1}{2}t$

① $c_1 = -c_2 = -\left(-\frac{1}{2}t\right) = \frac{1}{2}t$

Plugging this back into

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

gives

$$\left(\frac{1}{2}t\right)\vec{v}_1 - \left(\frac{1}{2}t\right)\vec{v}_2 + t\vec{v}_3 = \vec{0}$$

for any t.

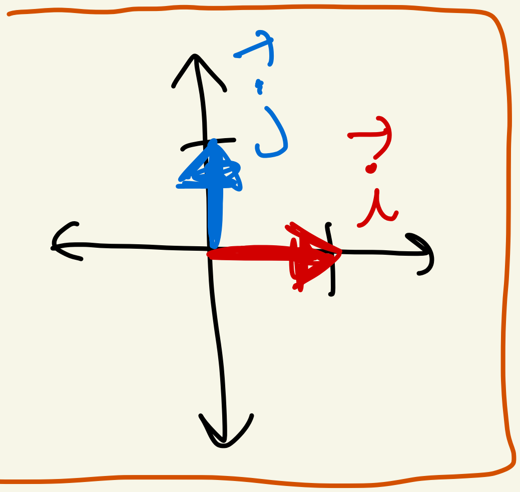
For example if $t=2$ we get

$$\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \vec{0}$$

So, $\vec{v}_1 = \vec{v}_2 - 2\vec{v}_3$

The vectors are linearly dependent.

Ex: Let $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ in $\mathbb{R}^2$. Are $\vec{i}, \vec{j}$ linearly dependent or linearly independent?



Let's solve

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

We get

$$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

This gives

$$\langle c_1, 0 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$$

Then,

$$\langle c_1, c_2 \rangle = \langle 0, 0 \rangle$$

So, $c_1 = 0$, $c_2 = 0$.

Thus, $\vec{i}, \vec{j}$ are linearly independent

Coordinate system theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n linearly independent vectors in $\mathbb{R}^n$.

Then, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are called a <u>coordinate system</u> or <u>basis</u> for $\mathbb{R}^n$. Moreover, if $\vec{v}$ is any vector in $\mathbb{R}^n$ then $\vec{v}$ can be expressed uniquely in the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

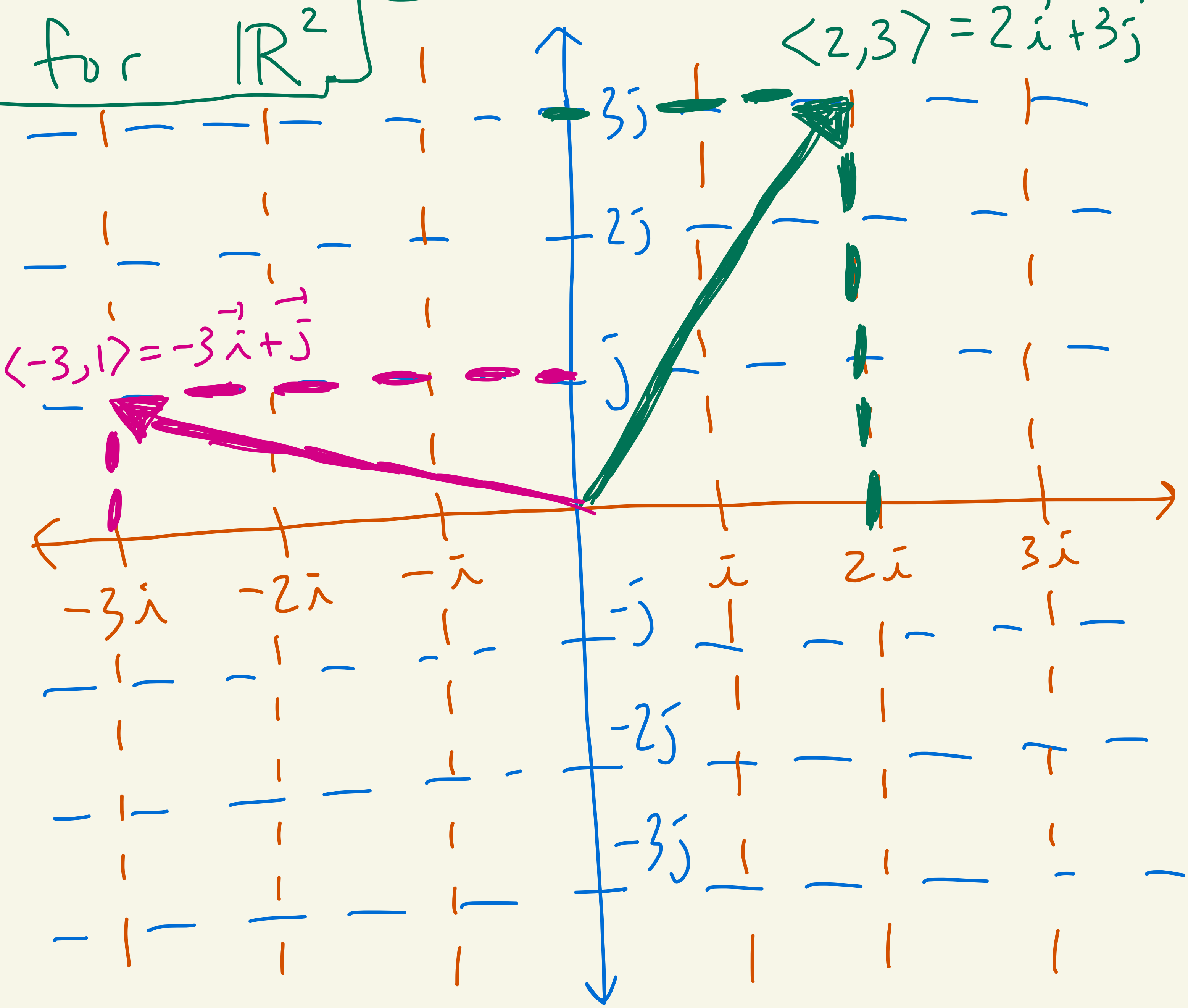
If we write $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ this means β is the name of our coordinate system and the brackets mean we've fixed an ordering on $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ then $c_1, c_2, \dots, c_n$ are called the <u>coordinates</u> of $\vec{v}$ with respect to β and we write

$$[\vec{v}]_\beta = \langle c_1, c_2, \dots, c_n \rangle$$

<u>Ex:</u> $\beta = [\vec{i}, \vec{j}]$ where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ in $\mathbb{R}^2$.

We have 2 linearly independent vectors in $\mathbb{R}^2$. So, β is a basis or coordinate system for $\mathbb{R}^2$.



$$\boxed{\beta = [\vec{i}, \vec{j}\,]}$$

$\vec{v} = \langle 2, 3 \rangle$

$\vec{v} = 2\vec{i} + 3\vec{j}$

$[\vec{v}]_\beta = \langle 2, 3 \rangle$

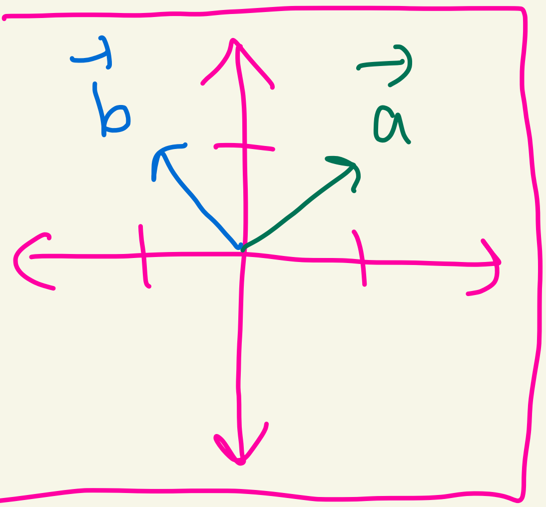
$\vec{w} = \langle -3, 1 \rangle$

$\vec{w} = -3\vec{i} + 1 \cdot \vec{j}$

$[\vec{w}]_\beta = \langle -3, 1 \rangle$

$\beta = [\vec{i}, \vec{j}\,]$ is called the <u>standard basis</u> or <u>standard coordinate system</u> for $\mathbb{R}^2$.

<u>Ex:</u> Let $\beta = [\vec{a}, \vec{b}\,]$ where



$\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$

Let's show $\vec{a}, \vec{b}$ are linearly independent and hence make β a basis or coordinate system.

Want to solve

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

We get

$$c_1 \langle 1, 1 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

This gives

$$\langle c_1 - c_2, c_1 + c_2 \rangle = \langle 0, 0 \rangle$$

We get

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

We get

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

So,

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_2 &= 0 \end{aligned}} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Thus,

(2) $c_2 = 0$

(1) $c_1 = c_2 = 0$

Thus, the only solution to

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$. So, $\vec{a}, \vec{b}$ are linearly independent.