Math 2550-01
11/18/24

Topic 8 – Linear Transformations and Eigenvalues

<u>Def</u>: Given two sets A and B a <u>function</u> f between A and B is a rule that assigns to each element x in A a unique element $f(x)$ in B.

We write $f : A \longrightarrow B$

name of function

domain or input to function

where outputs live

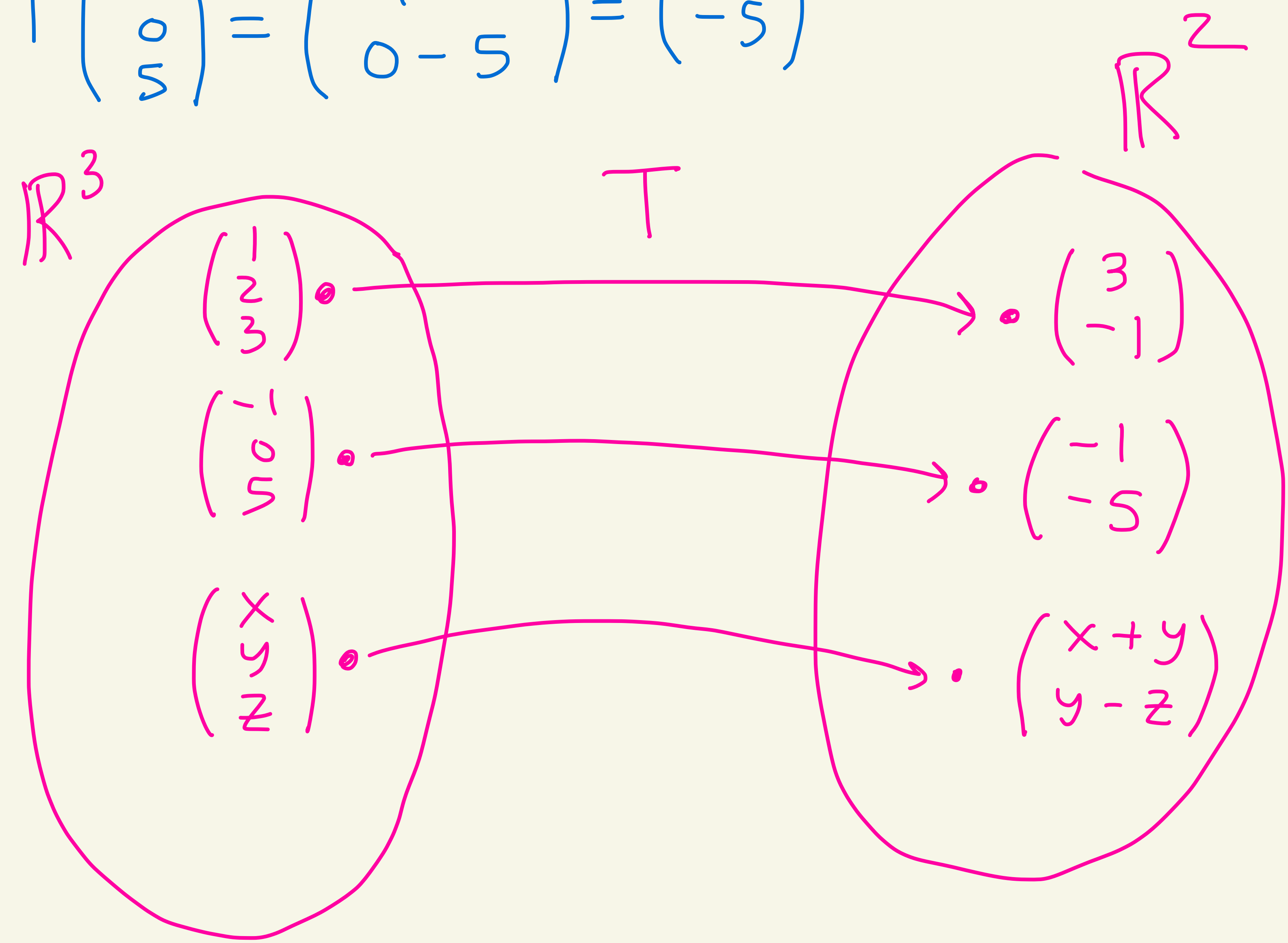
<u>Ex:</u> Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y-z \end{pmatrix}$$

For example:

$$T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1+0 \\ 0-5 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$



Note that

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\y-z\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 1 & 0\\0 & 1 & -1\end{pmatrix}}_{A}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

So,

$T(\vec{v}) = A\vec{v}$ where $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

<u>Def:</u> A <u>linear transformation</u> is a function $T: \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{v}) = A\vec{v}$ where A is an $m \times n$ matrix

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<u>Ex:</u> $T: \mathbb{R}^3 \to \mathbb{R}^2$

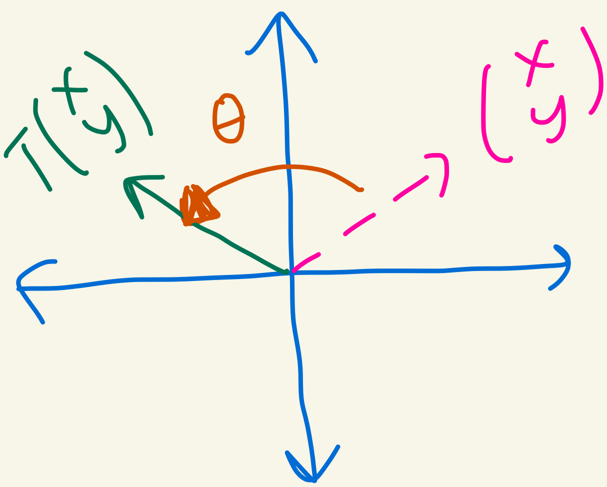
$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

given above is a linear transformation

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

This is a linear transformation that rotates $\begin{pmatrix} x \\ y \end{pmatrix}$ by θ degrees.



Rotate by $\theta = 90^\circ$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} -y \\ x \end{pmatrix}$$

<u>Theorem:</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a function. T is a linear transformation if and only if the following two properties hold for all vectors $\vec{v}$ and $\vec{w}$ and scalars α:

1. $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$
2. $T(\alpha\vec{v}) = \alpha T(\vec{v})$

We will now define what an eigenvalue/eigenvector are for a linear transformation with a square $n \times n$ matrix

Def: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $T(\vec{v}) = A\vec{v}$ where A is $n \times n$.
Suppose that $\vec{v}$ is a vector in $\mathbb{R}^n$ with

- $\vec{v} \neq \vec{0}$
- $T(\vec{v}) = \lambda\vec{v}$ $\quad \boxed{A\vec{v} = \lambda\vec{v}}$

where λ is a scalar.
Then λ is called an eigenvalue of T (or of A) and $\vec{v}$ is called an eigenvector of T (or of A) associated with λ.

If λ is an eigenvalue of T (or of A)

then define the eigenspace

$$E_\lambda(T) = \{ \vec{w} \mid \underbrace{T(\vec{w}) = \lambda \vec{w}}_{A\vec{w} = \lambda\vec{w}} \}$$

$\underbrace{E_\lambda(T)}$ we can also call this $E_\lambda(A)$

The eigenspace $E_\lambda(T)$ consists of all eigenvectors corresponding to λ and also the vector $\vec{0}$ to make $E_\lambda(T)$ a subspace

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Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that

$$T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \cdot 3 - 9 \cdot 2 \\ 4 \cdot 3 - 2 \cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 8 \end{pmatrix} = 4 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So,

$$T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 4 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{gathered} T(\vec{v}) = \lambda \vec{v} \\ \lambda = 4 \\ \vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \neq \vec{0} \end{gathered}}$$

$\lambda = 4$ is an eigenvalue with corresponding eigenvector $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$E_4(T) = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \dots \right\}$$

infinitely many more

How do we find the eigenvalues of a linear transformation T where $T(\vec{v}) = A\vec{v}$ and A is $n \times n$?

Suppose $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda\vec{v}$

(So, λ is an eigenvalue)

This gives $A\vec{v} - \lambda\vec{v} = \vec{0}$

This gives $\underbrace{(A - \lambda I_n)\vec{v}}_{\text{this is } A\vec{v} - \lambda I_n\vec{v} = A\vec{v} - \lambda\vec{v}} = \vec{0}$

We must have that $A - \lambda I_n$ has no inverse. Because if it did then we would get:

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

$$\underbrace{(A - \lambda I_n)^{-1}(A - \lambda I_n)\vec{v}}_{\vec{v}} = \underbrace{(A - \lambda I_n)^{-1}\vec{0}}_{\vec{0}}$$

giving $\vec{v} = \vec{0}$ which isn't the case since we started with $\vec{v} \neq \vec{0}$.

So, $(A-\lambda I_n)^{-1}$ does not exist.

So, $\det(A-\lambda I_n)=0$

Summary: The eigenvalues of T (or of A) are the λ that satisfy

$$\underbrace{\det(A-\lambda I_n)}_{\text{characteristic polynomial of } A \text{ or } T}=0$$