Math 2550-01
11/20/24

(Topic 8 continued...)

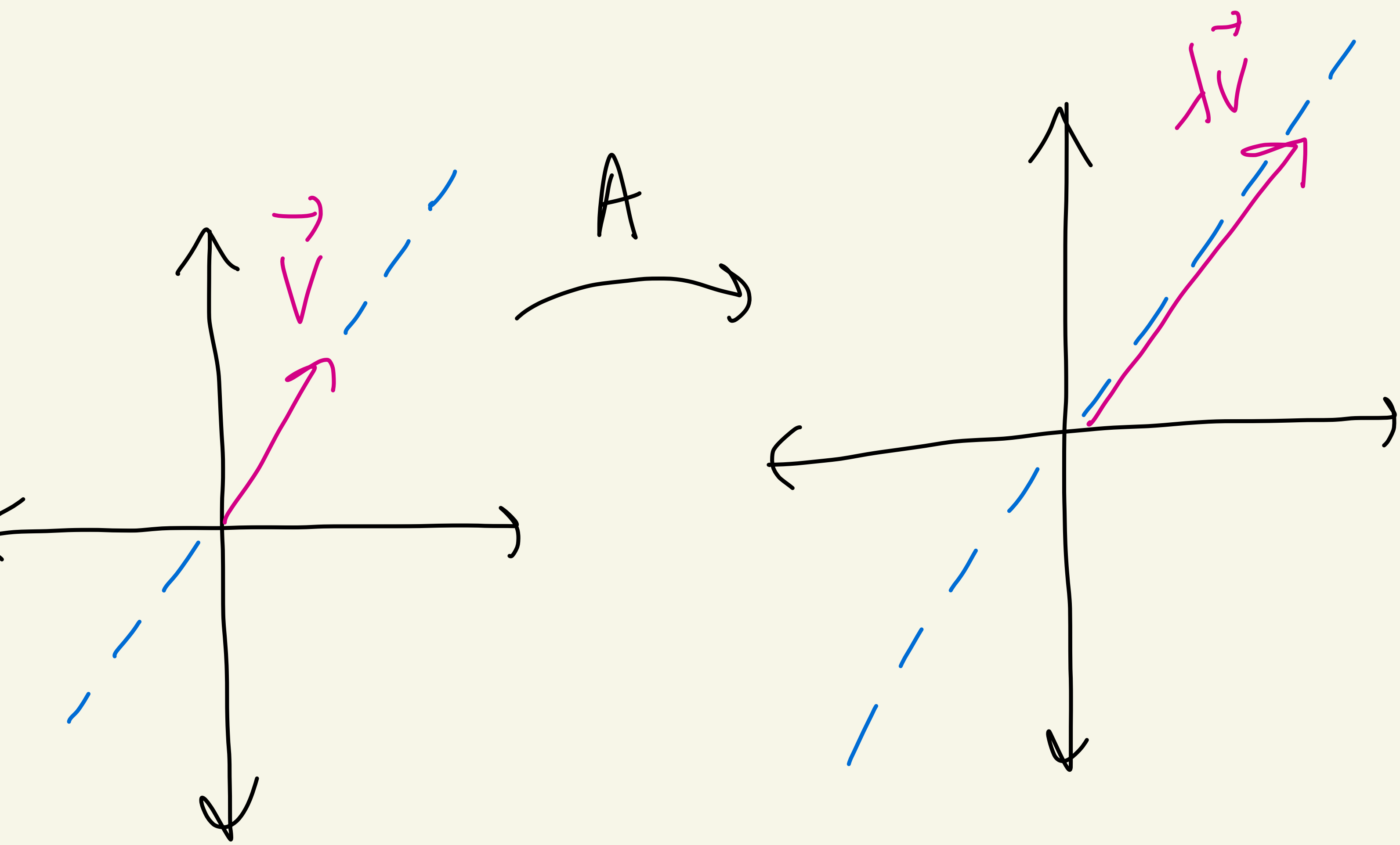
Recap from last time...

λ
lambda

$$A\vec{v} = \lambda\vec{v}$$
$$\vec{v} \neq \vec{0}$$

λ - eigenvalue

$\vec{v}$ - eigenvector



The eigenvalues of A are the solutions to $\det(A-\lambda I)=0$

END OF RECAP

Ex: Let $A=\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Let's find the eigenvalues of A.

We get

$\det(A-\lambda I) =$

$$= \det\left(\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}}_{A} - \lambda \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\right)$$

$$= \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) - (0)(8)$$

$$= (3-\lambda)(-1-\lambda)$$

Want $\det(A-\lambda I)=0$

which is $\underbrace{(3-\lambda)}_{\substack{3-\lambda=0 \\ \lambda=3}}\underbrace{(-1-\lambda)}_{\substack{-1-\lambda=0 \\ \lambda=-1}}=0$

$\boxed{\text{So the eigenvalues of } A \text{ are } \lambda=3,-1.}$

Let's find the eigenvectors for $\lambda=3$.

Want to solve $A\vec{v} = 3\vec{v}$.

Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

Want to solve $\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{v}} = 3 \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{v}}$

$$A\vec{v} = 3\vec{v}$$

This gives $\begin{pmatrix} 3x + 0y \\ 8x - y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

We gives $\boxed{\begin{aligned} 3x &= 3x \\ 8x - y &= 3y \end{aligned}}$

$\downarrow$

This gives $\boxed{\begin{aligned} 0 &= 0 \\ 8x - 4y &= 0 \end{aligned}}$

So:

$$\boxed{\begin{aligned} x - \tfrac{1}{2}y &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: x
free: y

Solutions: $\boxed{\begin{aligned} y &= t \\ x &= \tfrac{1}{2}y = \tfrac{1}{2}t \end{aligned}}$

The eigenvectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ associated with $\lambda = 3$ that solve $A\vec{v} = 3\vec{v}$ are of the form $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

that is they are the multiples of the vector $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.

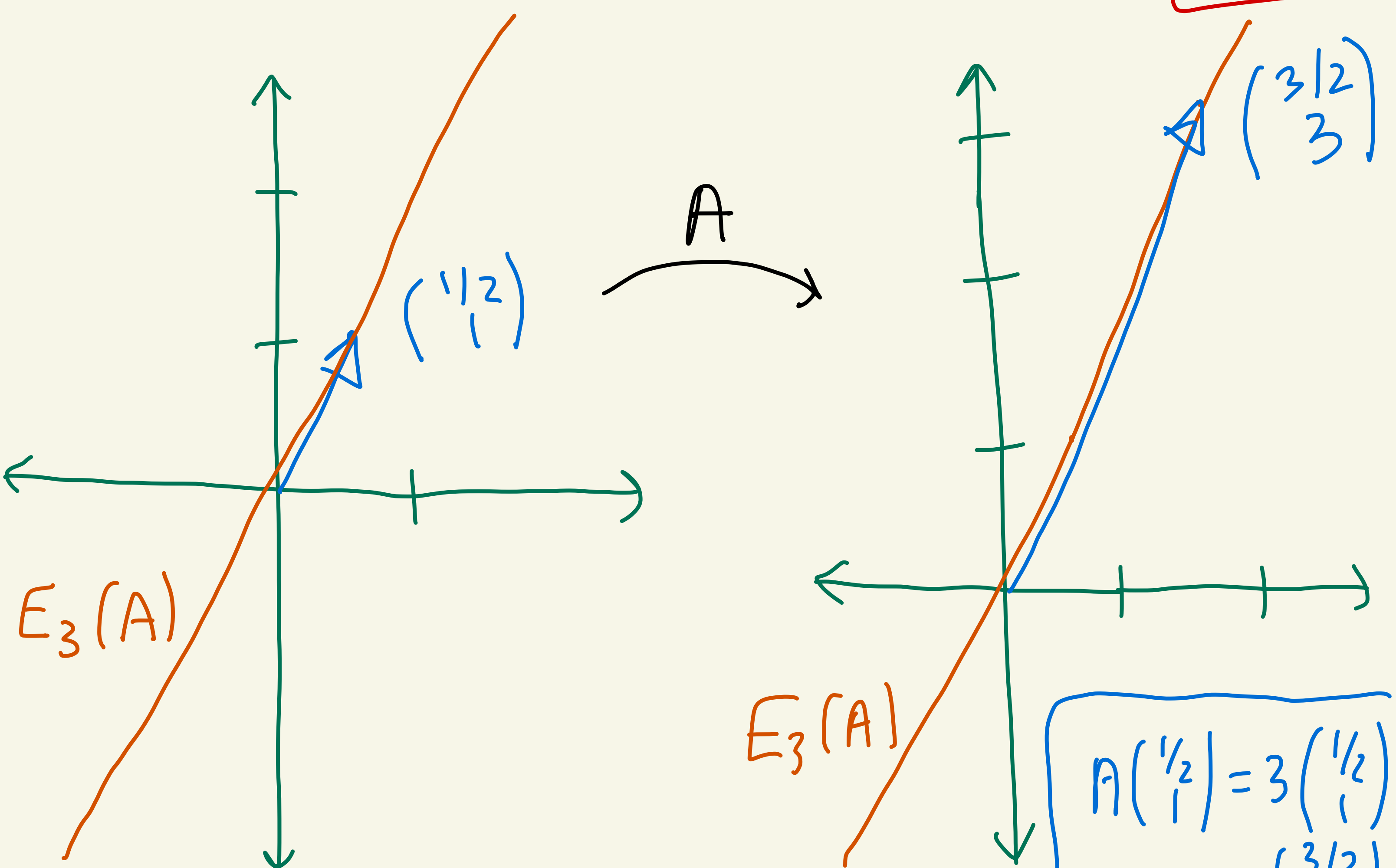
So, the eigenspace of $\lambda = 3$ is

$$E_3(A) = \left\{ \vec{v} \mid A\vec{v} = 3\vec{v} \right\}$$

$$= \left\{ t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}}_{t=1}, \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{t=2}, \underbrace{\begin{pmatrix} -1/2 \\ -1 \end{pmatrix}}_{t=-1}, \dots \right\}$$

$\boxed{\text{infinitely many}}$



$$\boxed{A\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \end{pmatrix}}$$

So, $\dim(E_3(A)) = 1$ with basis $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

Now let's find the eigenvectors that go with $\lambda = -1$.

Want to solve

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}}_{A\vec{v} = -\vec{v}}$$

This gives

$$\begin{pmatrix} 3x + 0y \\ 8x - y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} 3x &= -x \\ 8x - y &= -y \end{aligned}}$$

Or

$$\boxed{\begin{aligned} 4x &= 0 \\ 8x &= 0 \end{aligned}}$$

We get

$$\left(\begin{array}{cc|c} 4 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 8 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-8R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

giving

$$\boxed{\begin{aligned} x &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: x

free: y

Solution:

$$\boxed{\begin{aligned} x &= 0 \\ y &= t \end{aligned}}$$

So the eigenvectors $\vec{v}$ associated with $\lambda = -1$ that solve $A\vec{v} = -\vec{v}$ are of the form

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

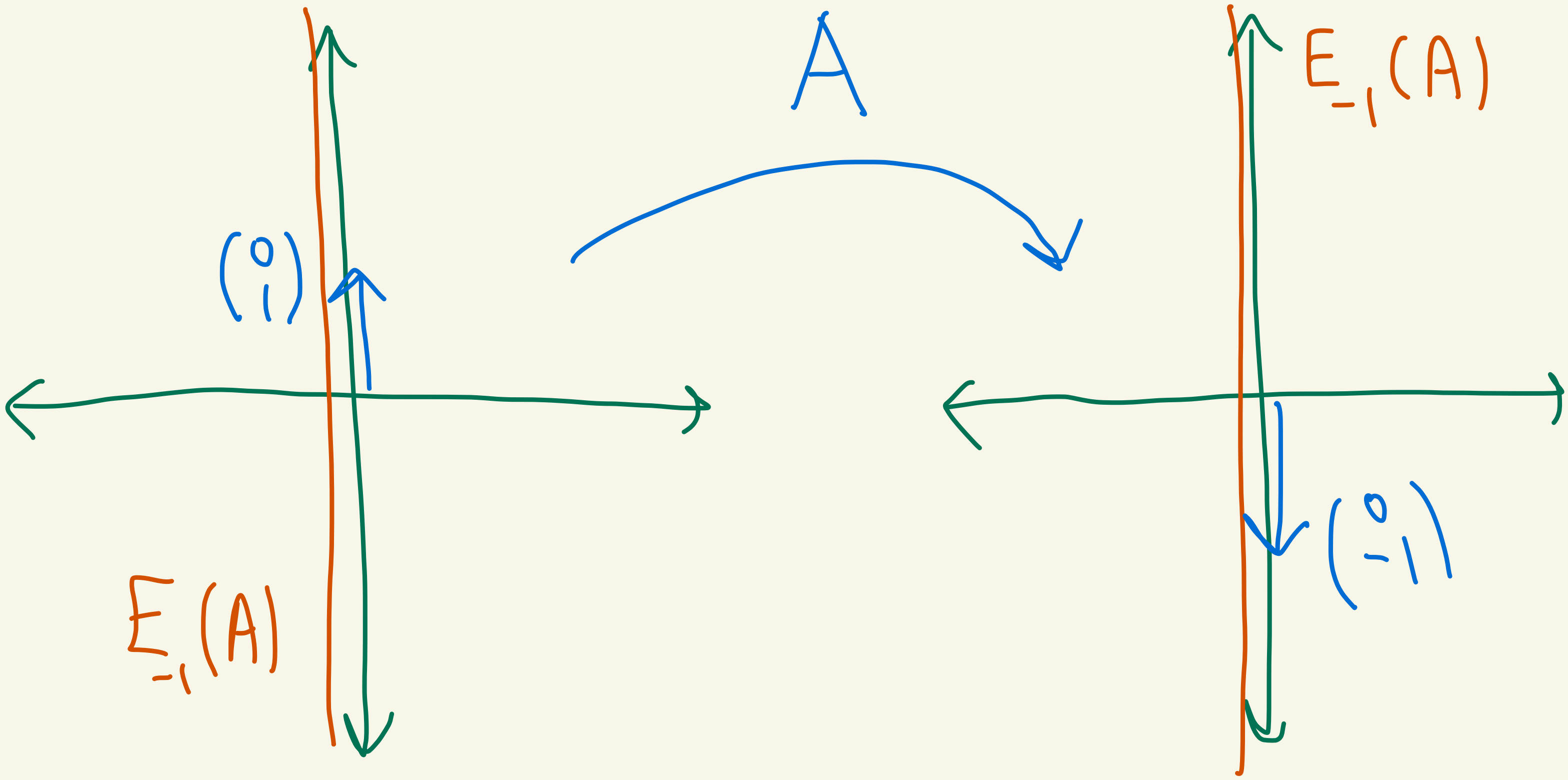
So the eigenspace is

$$E_{-1}(A) = \left\{ \vec{v} \mid A\vec{v} = -\vec{v} \right\}$$

$$= \left\{ t\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{t=1}, \underbrace{\begin{pmatrix} 0 \\ -2 \end{pmatrix}}_{t=-2}, \underbrace{\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}}_{t=\sqrt{2}}, \ldots \right\}$$

So, $\dim(E_{-1}(A)) = 1$ with basis $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



$$\boxed{A\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}}$$