Math 2550-01

11/4/24

- Finish Topic 7
- Review for test

(Topic 7 continued...)

Ex: In $\mathbb{R}^3$, let

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} x+y = 0 \\ \phantom{x+{}} y-5z = 0 \end{array} \right\}$$

By the homogeneous subspace theorem W will be a subspace of $\mathbb{R}^3$.

Let's find a basis for W.

Let $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in W.

Then, $\boxed{\begin{array}{l} x+y = 0 \\ \phantom{x+{}} y-5z = 0 \end{array}}$ $\begin{array}{l} ① \\ ② \end{array}$

already reduced

free variable is z

Solution is

$z = t$

2 $y = 5z = 5t$

1 $x = -y = -5t$

Thus,

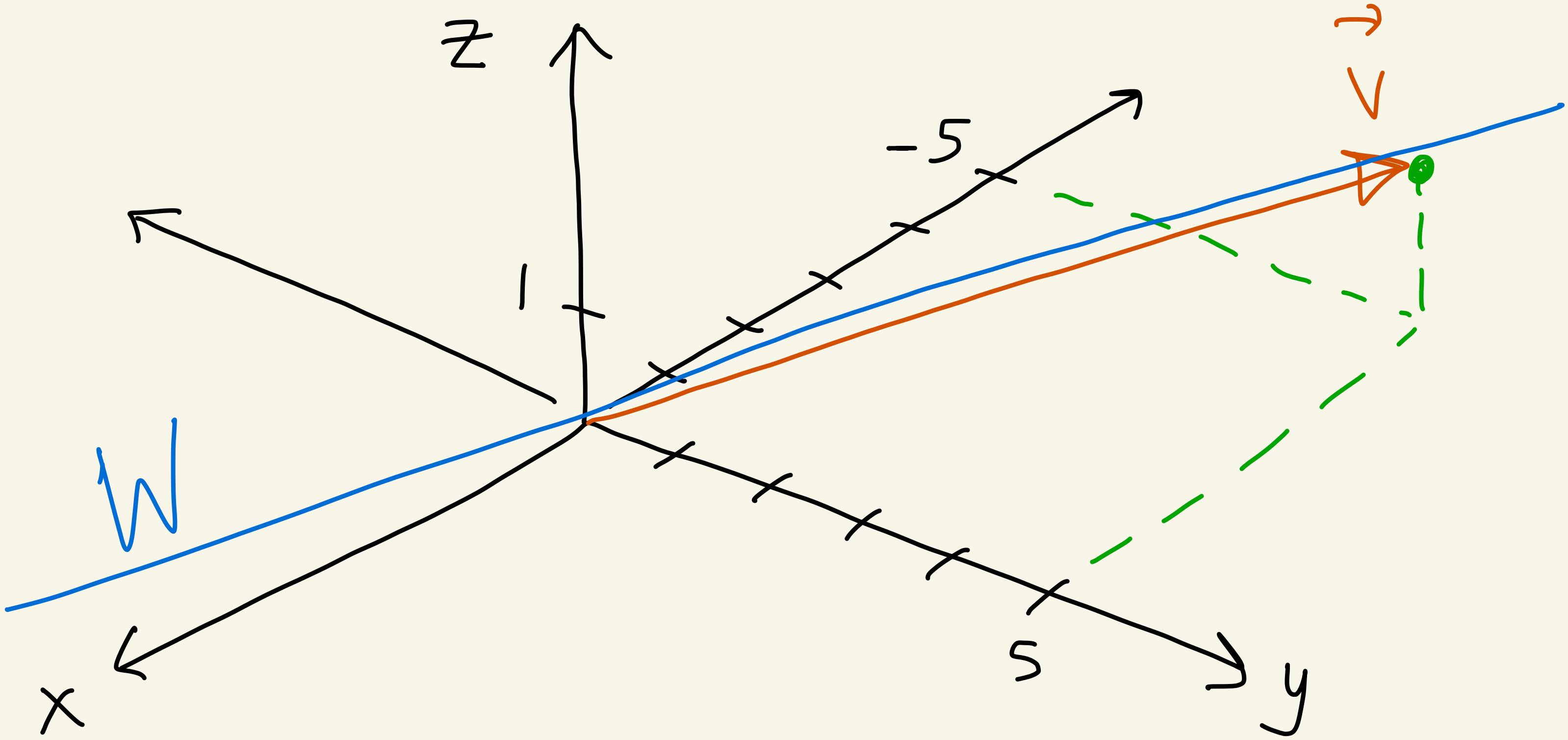
$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ 5t \\ t \end{pmatrix} = t\begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$$

Thus,

$$W = \text{span}\left(\begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}\right)$$

So, $\beta = \left[\begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}\right]$ is a basis for W. [β is a linearly independent set since it has one non-zero vector.]

$\dim(W) = 1$ since β has 1 vector



$$\boxed{\begin{array}{l} \vec{v} = \langle -5, 5, 1 \rangle \\ W = \text{span}(\vec{v}) \end{array}}$$

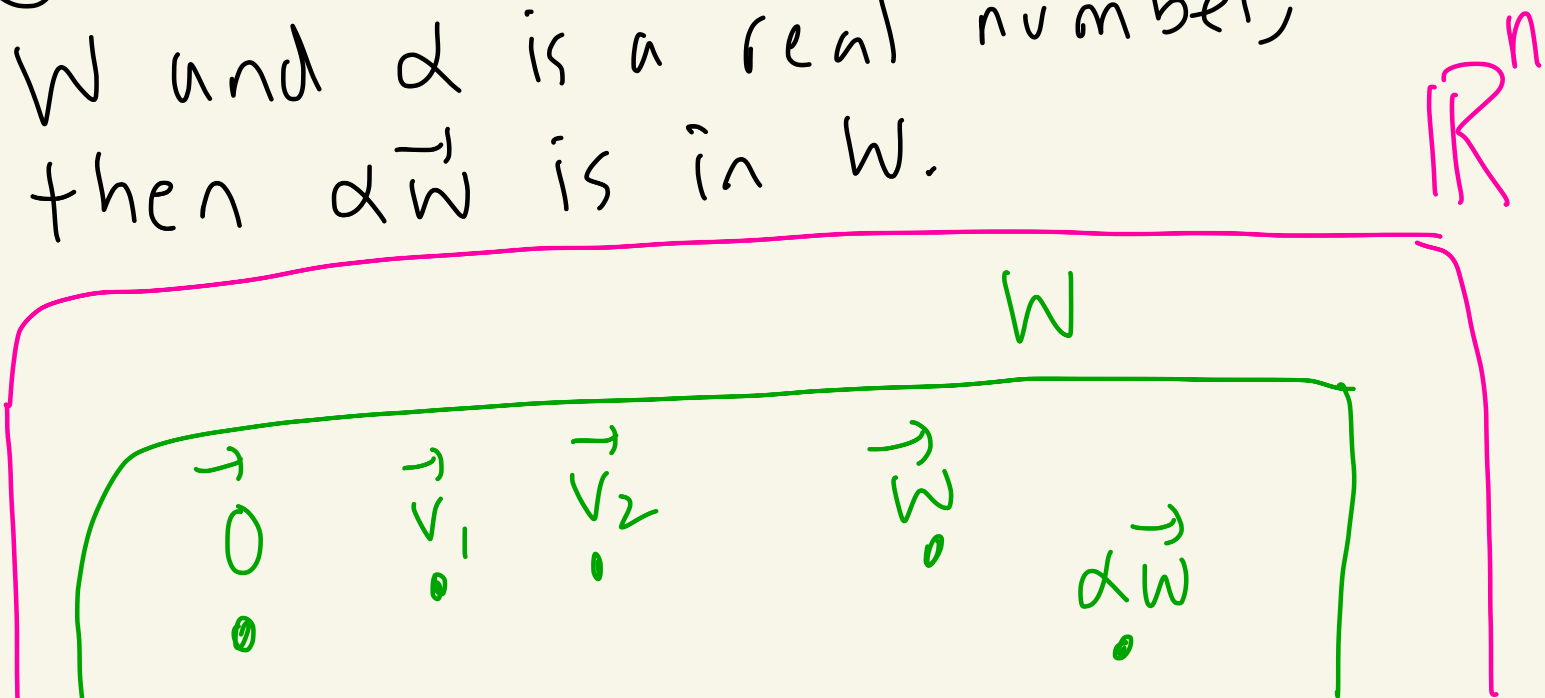
Theorem: (Dimension is well-defined)

Suppose W is a subspace of $\mathbb{R}^n$.
Let $\beta_1 = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_a]$ and $\beta_2 = [\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_b]$ be two bases for W. Then, $a = b$.

Here is how people usually define subspace

Theorem: Let W be a subset of $\mathbb{R}^n$. Then, W is a subspace if and only if 3 conditions hold:

1. $\vec{0}$ is in W
2. (closure under addition) If $\vec{v}_1, \vec{v}_2$ are in W, then $\vec{v}_1 + \vec{v}_2$ is in W
3. (closure under scaling) If $\vec{w}$ is in W and α is a real number, then $\alpha\vec{w}$ is in W.



$\vec{v}_1 + \vec{v}_2$

0

$\boxed{\text{HW 4} \\ \text{1(a)}}$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Determine if A and B are inverses of each other.

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1)+(-1)(0) & (1)(\frac{1}{2})+(-1)(\frac{1}{2}) \\ (0)(1)+(2)(0) & (0)(\frac{1}{2})+(2)(\frac{1}{2}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

So, A and B are inverses.

HW 4 3(a)

Find A^{-1} if it exists.

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}$$

$$\left(\underbrace{\begin{matrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{matrix} 1 & 0 & 3 \\ 3 & 4 & -1 \\ 2 & 5 & -4 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left(\begin{matrix} 1 & 0 & 3 \\ 0 & 4 & -10 \\ 0 & 5 & -10 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{matrix}\right)$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{-5R_2 + R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 5/2 & -5/4 & 7/4 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{2}{5}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right)$$

$$\xrightarrow[\frac{5}{2}R_3 + R_2 \to R_2]{-3R_3 + R_1 \to R_1} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \middle| \underbrace{\begin{matrix} 3/2 & \frac{-11}{10} & \frac{-6}{5} \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{matrix}}_{A^{-1}}\right)$$

So, $A^{-1} = \begin{pmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{pmatrix}$

$\boxed{\text{HW 4}}$

5 (c) Solve

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

by inverting the coefficient matrix

Convert the system into a matrix equation:

$$\underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}}_{\vec{b}} \qquad (*)$$

Suppose you know $A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}$

Idea:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Multiply on left of $(*)$ by A^{-1}

$$\cancel{\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}}_{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}}_{\vec{b}}$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} \underbrace{=}_{=} \underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}}_{\vec{b}}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (-1)(4)+(0)(-1)+(1)(3) \\ (0)(4)+(-1)(-1)+(1)(3) \\ (2)(4)+(3)(-1)+(-4)(3) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}}$$

Answer