Math 2550-01

12/2/24

Topic 9 – Matrices of linear transformations

Sometimes you have a linear transformation whose input and output is expressed in the usual xy or xyz coordinate system, but you instead want the input and output to be expressed in terms of a different coordinate system. We will learn how to do this.

__Def:__ Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ be a basis/coordinate system for $\mathbb{R}^n$.

The matrix

$$\underbrace{[T]_\beta}_{\text{notation for matrix}} = \left([T(\vec{v}_1)]_\beta \;\middle|\; [T(\vec{v}_2)]_\beta \;\middle|\; \cdots \;\middle|\; [T(\vec{v}_n)]_\beta \right)$$

columns of matrix

is called the __matrix for T with respect to β.__

What does $[T]_\beta$ do?

For any vector $\vec{v}$ we will get

$$\underbrace{[T(\vec{v})]_\beta}_{T(\vec{v})\text{'s } \beta\text{-coordinates}} = \underbrace{[T]_\beta}_{\text{matrix above}} \underbrace{[\vec{v}]_\beta}_{\vec{v}\text{'s } \beta\text{-coordinates}}$$

So, $[T]_\beta$ computes T but it wants β-coordinates as input and it outputs β-coordinates.

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

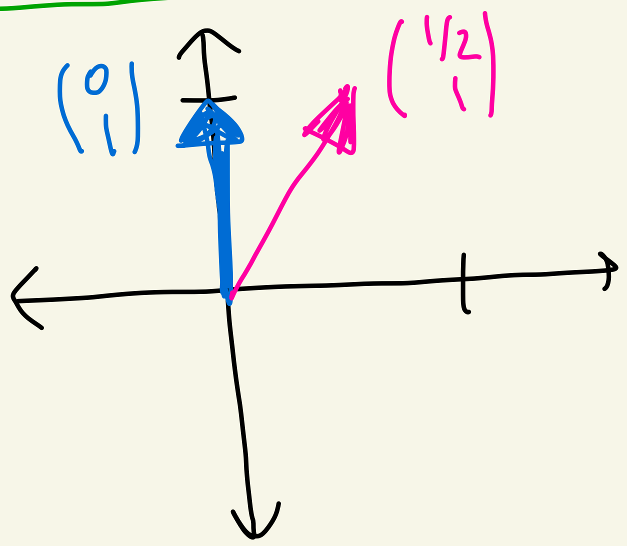
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 8x-y \end{pmatrix}$$

Let

$$\beta = \left[\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

You can check these vectors are linearly independent so β is a basis for $\mathbb{R}^2$.

eigenvector: $T(\vec{v}) = \lambda \vec{v}$



Note: I picked the eigenvectors of T.

Recall

$$T\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's find $[T]_\beta$

$$T\begin{pmatrix}1/2\\1\end{pmatrix} = \begin{pmatrix}3/2\\3\end{pmatrix} = 3\cdot\begin{pmatrix}1/2\\1\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = 0\cdot\begin{pmatrix}1/2\\1\end{pmatrix} - 1\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

plug β into T

express the answers in β-coordinates

Thus,

$$[T]_\beta = \left(\left[T\begin{pmatrix}1/2\\1\end{pmatrix}\right]_\beta \;\middle|\; \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta \right)$$

$$= \begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}$$

What does this matrix do?

Let $\vec{v} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

Then, $T(\vec{v}) = T\begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 6 \\ 22 \end{pmatrix}$

$$\boxed{T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 8x-y \end{pmatrix}}$$

xy coordinate system calculation or the $\vec{i}, \vec{j}$ coordinate system calculation

Let's use β instead.

Note: $\vec{v} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} = 4\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - 10\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $[\vec{v}]_\beta = \begin{pmatrix} 4 \\ -10 \end{pmatrix}$

And

$$[T]_\beta[\vec{v}]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ -10 \end{pmatrix} = \underbrace{\begin{pmatrix} 12 \\ 10 \end{pmatrix}}_{[T(\vec{v})]_\beta}$$

So, it should be that

$$T(\vec{v}) = 12\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + 10\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Which gives $T(\vec{v}) = \begin{pmatrix} 6 \\ 22 \end{pmatrix}$

which is what we had above.

<u>Summary:</u> $\beta = \left[\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

Given $\vec{v}$ write $\vec{v} = c_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $[\vec{v}]_\beta = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Compute

$$[T]_\beta [\vec{v}]_\beta = \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Computes } T} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\beta\text{-coordinate input}} = \overbrace{\underbrace{\begin{pmatrix} 3c_1 \\ -c_2 \end{pmatrix}}_{\text{this is } [T(\vec{v})]_\beta}}^{\beta\text{-coordinate output}}$$

This expresses the identity:

$$T\left(c_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 3c_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using eigenvectors like this is called "diagonalizing T"