Math 2550-01
8/28/24

Topic 1 continued...

<u>Def:</u> Let

$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and

$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$

be in $\mathbb{R}^n$.

Define the <u>dot product</u> to be

$$\vec{v} \cdot \vec{w} = \underbrace{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}_{\text{this is a number}}$$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, 5 \rangle$ and $\vec{w} = \langle -5, 4 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 2, 5 \rangle \cdot \langle -5, 4 \rangle$$

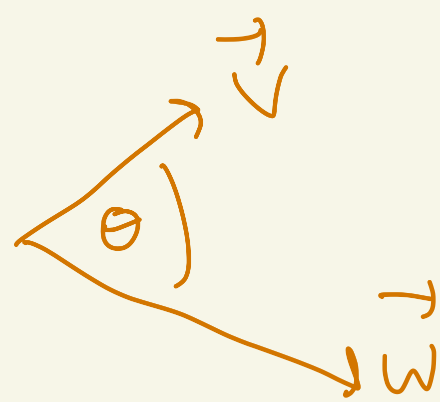
$$= (2)(-5) + (5)(4)$$

$$= -10 + 20$$

$$= 10$$

In Calculus (in $\mathbb{R}^2$ and $\mathbb{R}^3$)

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|},$$



Ex: In $\mathbb{R}^5$ we have

$$\langle 1, 2, -1, 0, 3 \rangle \cdot \langle \tfrac{1}{2}, 1, -3, 4, 10 \rangle$$

$$= (1)\left(\tfrac{1}{2}\right) + (2)(1) + (-1)(-3) + (0)(4) + (3)(10)$$

$$= \tfrac{1}{2} + 2 + 3 + 0 + 30$$

$$= 35.5$$

Properties of the dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$

Let α be a $\underbrace{\text{scalar}}_{\text{number}}$ in $\mathbb{R}$

Then:

① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

③ $\alpha(\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha \vec{v})$

Ex of ③:

$5(\vec{u} \cdot \vec{v}) = (5\vec{u}) \cdot \vec{v} = \vec{u} \cdot (5\vec{v})$

$\alpha = 5$

proof of 2 when $n=3$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^3$.

Then,

$\vec{u} = \langle a,b,c \rangle$, $\vec{v} = \langle d,e,f \rangle$, $\vec{w} = \langle g,h,i \rangle$

where a,b,c,d,e,f,g,h,i are numbers.

We have

$$\vec{u} \cdot (\vec{v} + \vec{w}) =$$

$$= \langle a,b,c \rangle \cdot \left(\langle d,e,f \rangle + \langle g,h,i \rangle \right)$$

$$= \langle a,b,c \rangle \cdot \langle d+g, e+h, f+i \rangle$$

$$= a(d+g) + b(e+h) + c(f+i)$$

$$= ad + ag + be + bh + cf + ci$$

EQUAL

Also, we have

$$\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w} =$$

$$= \underbrace{\langle a,b,c\rangle\cdot\langle d,e,f\rangle}_{ad+be+cf} + \underbrace{\langle a,b,c\rangle\cdot\langle g,h,i\rangle}_{ag+bh+ci}$$

$$= ad+be+cf + ag+bh+ci$$

Thus,

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$

$\square$

HW 1 - Part 1

#10 List 3 elements from the set

$$S = \{ c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle \mid c_1, c_2 \in \mathbb{R} \}$$

read: S consists of all the elements of the form $c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$ where c_1, c_2 are real numbers

For example if $c_1 = 1$ and $c_2 = 3$ then we get

$c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$

$= 1 \cdot \langle 1,1,1 \rangle + 3 \langle 0,0,5 \rangle$

$= \langle 1,1,1 \rangle + \langle 0,0,15 \rangle$

$= \langle 1,1,16 \rangle$

So, $\langle 1,1,16 \rangle$ is in S

If $c_1 = 0$ and $c_2 = 0$, then

$c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$

$= 0 \langle 1,1,1 \rangle + 0 \langle 0,0,5 \rangle$

$= \langle 0,0,0 \rangle + \langle 0,0,0 \rangle$

$= \langle 0,0,0 \rangle$

So, $\langle 0,0,0 \rangle$ is in S

If $c_1 = 1$ and $c_2 = 0$, then

we get

$1 \cdot \langle 1,1,1 \rangle + 0 \cdot \langle 0,0,5 \rangle$

$= \langle 1, 1, 17 \rangle + \langle 0, 0, 0 \rangle$

$= \langle 1, 1, 17 \rangle.$

So, $\langle 1, 1, 17 \rangle$ is in S.

So,

$S = \{ \langle 1, 1, 16 \rangle, \langle 0, 0, 0 \rangle, \langle 1, 1, 17 \rangle, \ldots \}$

infinitely many more

Topic 2 - Matrices

<u>Def:</u> A <u>matrix</u> is a rectangular array of numbers. If M is a matrix with m rows and n columns, then we say that M is $m \times n$.

read: "m by n"

Abstractly you can write an $m \times n$ matrix as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Where a_{ij} is the entry in row i and column j.

Ex:

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

M is $\underbrace{2}_{\boxed{\text{2 rows}}} \times \underbrace{2}_{\boxed{\text{2 columns}}}$

$$\boxed{\begin{aligned} a_{11} &= 0 \\ a_{12} &= 1 \\ a_{21} &= 2 \\ a_{22} &= 10 \end{aligned}}$$

Ex:

$$A = \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}$$

A is $\underbrace{1}_{\boxed{\text{1 row}}} \times \underbrace{3}_{\boxed{\text{3 columns}}}$.

$$\boxed{\begin{aligned} a_{11} &= 1 \\ a_{12} &= 5 \\ a_{13} &= 3 \end{aligned}}$$

Ex:

$$B = \begin{pmatrix} 1 & 3 & 5 & 4 & 2 \\ 0 & 2 & 7 & 6 & -1 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 3 & 7 & 9 & 7 \end{pmatrix}$$

$a_{24} = 6$

$a_{42} = 3$

B is $\underset{\text{4 rows}}{4} \times \underset{\text{5 columns}}{5}$

You can think of a vector as a matrix, either as a row or a column.

For example, $\vec{v} = \langle 1, 2, 3 \rangle$

You can think of $\vec{v}$ as:

$$\boxed{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \text{ 1x3 matrix}} \quad \text{or} \quad \boxed{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \leftarrow \text{3x1 matrix}}$$

<u>Def:</u> Let A and B be $m \times n$ matrices. [So A and B have the same dimensions.]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Define:

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

$$A-B = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

If α is a scalar/number, then

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$