Math 2550-01
9/16/24

Fact: Applying an elementary row operation to a system of linear equations doesn't change the solution space

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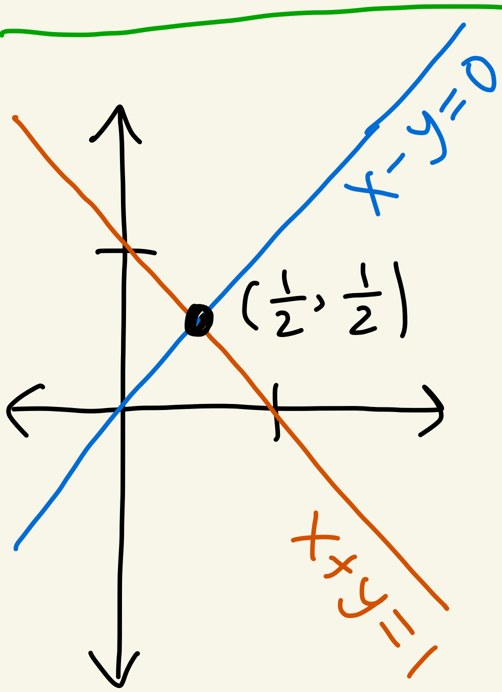
Ex:



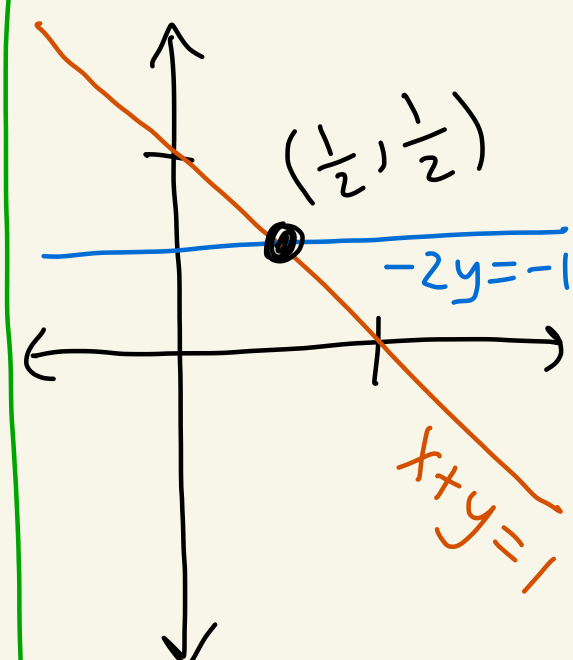
system:

$$\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}}$$

solutions:

$(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$

$$\xrightarrow{-R_1+R_2\to R_2}$$



system:

$$\boxed{\begin{aligned} x+y&=1 \\ -2y&=-1 \end{aligned}}$$

solutions:

$(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$

$$\boxed{\begin{array}{rl} -x-y=-1 & \leftarrow \boxed{-R_1} \\ +\quad x-y=0 & \leftarrow \boxed{R_2} \\ \hline -2y=-1 & \leftarrow \boxed{\text{new } R_2} \end{array}}$$

Def: If a row of a matrix does not consist entirely of zeros then the <u>leading entry</u> in that row is the first non-zero entry when scanning from left to right.

Ex:

$$A = \begin{array}{c} \begin{pmatrix} 0 & 5 & 2 & 0 \\ 1 & -1 & 10 & 1/2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \leftarrow \text{row 3} \\ \leftarrow \text{row 4} \end{array}$$

leading entry in row 1 is 5
leading entry in row 2 is 1
leading entry in row 3 is 3
no leading entry in row 4.

Def: A matrix is in **row echelon form** if the following are true:

① if there are any rows consisting entirely of zeros, then they are at the bottom of the matrix,

② in any two consecutive rows that do not consist entirely of zeros, then the leading entry in the lower row is to the right of the leading entry in the upper row, ③ if a row doesn't consist entirely of zeros, then its leading entry is 1.

Ex:

$$A = \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

(leading entries circled)

① ✓
② ✓
③ X

not in row echelon form

Ex:

$$A = \begin{pmatrix} 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① ✓
② ✓
③ ✓

(leading entries circled)

In row echelon form

Ex:

$$A = \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 1 & 5 & -3 \end{pmatrix}$$

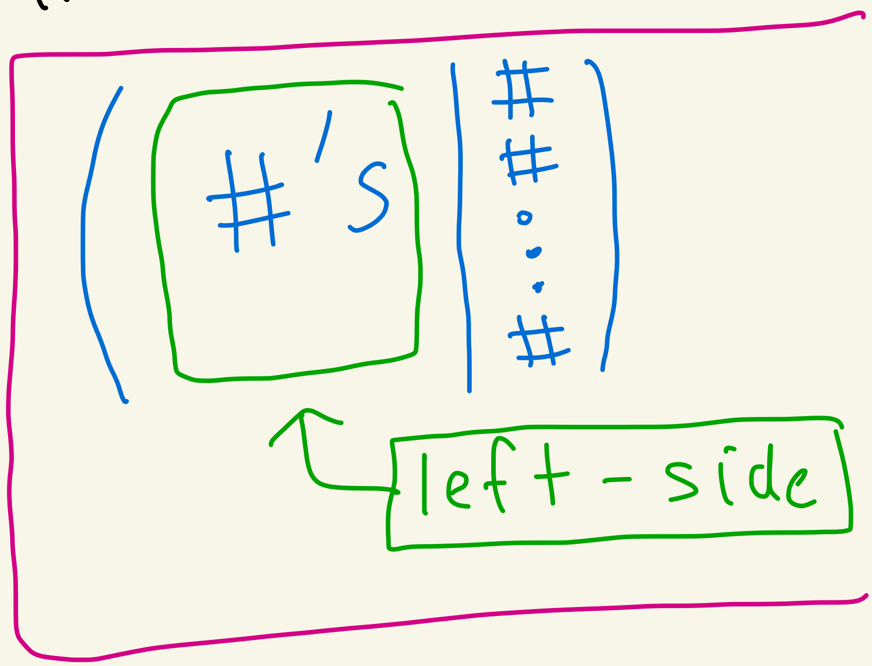
1 ✓

2 X

3 ✓

(leading entries circled)

Def: Suppose you have an augmented matrix for a system of linear equations.
Suppose you use elementary row operations to put the left-side of the matrix into row echelon form.



The variable corresponding to the leading entry of a row is called a **leading variable**.

Any variable that doesn't occur as a leading variable is called a **free variable**.

Ex: Suppose

$$\left(\begin{array}{ccc|c} 1 & 2 & 10 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

left-side is in row echelon form

corresponds to

$$\boxed{\begin{aligned} x + 2y + 10z &= -2 \\ z &= 5 \end{aligned}}$$

leading variables: x, z

free variable: y

Ex: Suppose

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

in row echelon form

corresponds to

$$\boxed{\begin{aligned} a - 2b \phantom{+ \tfrac{1}{2}c} &= 2 \\ b + \tfrac{1}{2}c &= 1 \\ c &= 5 \end{aligned}}$$

leading variables: a, b, c

free variables: none

Method to solve a system of linear equations (Gaussian elimination)

① Use elementary row operations to put the left side of the augmented matrix of the system into row echelon form.

② Look at the equations that go with this reduced system.

<u>case (a):</u> If one of the

equations is $0 = c$, where $c \neq 0$, then the system has no solutions.

<u>case (b):</u> If case (a) doesn't occur then we "back substitute" to solve the system as follows:

(i) Solve each equation for the leading variable.

(ii) Assign the free variables a new name

(iii) Beginning with the bottom/last equation, successively substitute each equation into the equation above it.