Math 2550-01

9/18/24

Ex: Solve

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

We have

want a 1 here ✓

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Use the 1 to make these 0

$$\begin{array}{rl} \left(\begin{array}{ccc|c} -2 & -2 & -4 & -18 \end{array}\right) & \leftarrow \boxed{-2R_1} \\ +\left(\begin{array}{ccc|c} 2 & 4 & -3 & 1 \end{array}\right) & \leftarrow \boxed{R_2} \\ \hline \left(\begin{array}{ccc|c} 0 & 2 & -7 & -17 \end{array}\right) & \leftarrow \boxed{\text{new } R_2} \end{array}$$

$$\xrightarrow{-3R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right)$$

use the 1 to make this 0

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -\frac{3}{2} \end{array}\right)$$

$$\frac{21}{2} - 11 = \frac{21-22}{2} = -\frac{1}{2}$$

$$\frac{51}{2} - 27 = \frac{51-54}{2} = \frac{-3}{2}$$

make this a 1

$$\xrightarrow{-2R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

is in row echelon form

Write the equations down:

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

① ② ③

leading variables: x, y, z

free variables: none

Solve each eqn for leading variables:

$$\begin{aligned} x &= 9 - y - 2z \\ y &= -\tfrac{17}{2} + \tfrac{7}{2}z \\ z &= 3 \end{aligned}$$

① ② ③

Back substitute:

③ $z = 3$

② $y = -\frac{17}{2} + \frac{7}{2}z = -\frac{17}{2} + \frac{7}{2}(3) = 2$

① $x = 9 - y - 2z = 9 - 2 - 2(3) = 1$

Answer: $x = 1, y = 2, z = 3$

Ex: Solve

$$\boxed{\begin{aligned} -2b+3c&=1\\ 3a+6b-3c&=-2\\ 6a+6b+3c&=5 \end{aligned}}$$

Want a 1 here

We get

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

use the 1 to make these 0

$$-6R_1+R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right)$$

make this a 1

$$-\tfrac{1}{2}R_2 \to R_2 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 9 \end{array}\right)$$

use the 1 to make this 0

$$6R_2+R_3 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

in row echelon form

Write down the equations:

$$\boxed{\begin{aligned} a + 2b - c &= -2/3 \\ b - \tfrac{3}{2}c &= -1/2 \\ 0 &= 6 \end{aligned}}$$

$\leftarrow \boxed{0a + 0b + 0c = 6}$

There are no solutions to the system since $0 = 6$ is impossible.

Ex: Solve

$$\begin{aligned} 5x_1 - 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 1 \end{aligned}$$

We have

want a 1 here

$$\left(\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{2R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ -2 & 1 & 3 & 1 \end{array}\right)$$

make this 0

$$\xrightarrow{2R_1+R_2\to R_2}\left(\begin{array}{ccc|c} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{array}\right)$$

this is in row echelon form

Write down the equations:

$$\boxed{\begin{aligned} x_1 \quad\quad + 12x_3 &= 2 \quad (1) \\ x_2 + 27x_3 &= 5 \quad (2) \end{aligned}}$$

leading variables x_1, x_2

free variables x_3

Solve for leading and give free variable a new name:

$$\boxed{\begin{aligned} x_1 &= 2-12x_3 \quad (1) \\ x_2 &= 5-27x_3 \quad (2) \\ x_3 &= t \quad (3) \end{aligned}}$$

Back-substitute:

(3) $x_3 = t$

② $x_2 = 5 - 27x_3 = 5 - 27t$

① $x_1 = 2 - 12x_3 = 2 - 12t$

Answer:

$x_1 = 2 - 12t$
$x_2 = 5 - 27t$
$x_3 = t$

where t can be any number

This would give you an infinite # of solutions to the system, one for each t. For example:

$t=0$: $x_1 = 2$, $x_2 = 5$, $x_3 = 0$

$t=1$: $x_1 = -10$, $x_2 = -22$, $x_3 = 1$

$t=10$: $x_1 = -118$, $x_2 = -265$, $x_3 = 10$

and so on...