Math 2550-01

9/25/24

<u>Theorem</u>: (Inverses are unique)
Let A be an $n \times n$ matrix that is invertible [so A has an inverse]. Then there is only <u>one</u> $n \times n$ matrix B where

$$AB = BA = I_n.$$

<u>Notation</u>: If A has an inverse, then we denote it by A^{-1}.

<u>Ex</u>: Last time we saw that

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$\boxed{\text{Procedure to find } A^{-1} \text{ if it exists}}$

Let A be an $n \times n$ matrix.

Start with the matrix

$$\left(A \mid I_n \right)$$

Apply elementary row operations until either (i) the left side becomes I_n, or (ii) the left side has a row of zeros.

If (i) happens, then A has an inverse and it will be on the right side of the matrix

If (ii) then A has no inverse, A^{-1} doesn't exist.

Ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Let's find the inverse we saw on Monday.

$$\left(\begin{array}{cc|cc} \overbrace{1 \quad\quad 1}^{A} & & \overbrace{1 \quad\quad 0}^{I_2} & \\ \underbrace{2 \quad\quad 1} & & 0 & 1 \end{array}\right)$$

we want to turn this side into I_2 if we can.

make this 0

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

make this 0

$$\xrightarrow{-R_2 + R_1 \to R_1} \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} & \underbrace{\begin{matrix} -1 & 1 \\ 2 & -1 \end{matrix}}_{A^{-1}} \end{array}\right)$$

So, A is invertible and

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Ex: Let $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$.

Find A^{-1} if it exists.

$$\left(\begin{array}{cc|cc} \overbrace{\begin{matrix} 1 & 5 \\ -2 & -10 \end{matrix}}^{A^{-1}} & & \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I_2} & \end{array}\right)$$

try to turn this side into I_2

make this 0

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

row of 0's on left side

So, $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$ does not have an inverse.

Ex: Find A^{-1} if it exists

When $A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{3 \quad 0 \quad 3}}^{A}} & \multicolumn{3}{c}{\overbrace{\hphantom{1 \quad 0 \quad 0}}^{I_3}} \\ 3 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

put 1 here

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[2R_1+R_3\to R_3]{-3R_1+R_2\to R_2}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-\frac{1}{3}R_2\to R_2}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make these 0

$$\xrightarrow[-5R_2+R_3\to R_3]{-R_2+R_1\to R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & -1 & \frac{5}{3} & -3 & 1 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & 1 & -5/3 & 3 & -1 \end{array}\right)$$

make these 0

$$\xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 4/3 & -2 & 1 \\ 0 & 0 & 1 & -\frac{5}{3} & 3 & -1 \end{array}\right)$$

I_3 $\qquad$ A^{-1}

So, A^{-1} exists and

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Theorem Let A and B be $n \times n$ matrices that are both invertible. [So A^{-1} and B^{-1} exist.]

Then:

1. AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$

2. A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T$$

Sometimes you can solve a system using inverses. To do this the first thing is to convert the system into a matrix equation.

Given

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}} \quad (*)$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then (*) is equivalent to $\boxed{A\vec{x} = \vec{b}}$

<u>Ex:</u> Consider $\boxed{\begin{aligned} 2x - y &= 3 \\ 5x + 10y &= 6 \end{aligned}}$

Let $A = \begin{pmatrix} 2 & -1 \\ 5 & 10 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Then

$$A\vec{x} = \vec{b}$$

becomes

$$\begin{pmatrix} 2 & -1 \\ 5 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

which is

$$\begin{pmatrix} 2x - y \\ 5x + 10y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

this is the same as

$$\boxed{\begin{aligned} 2x - y &= 3 \\ 5x + 10y &= 6 \end{aligned}}$$